Chapter 9 Complex numbers

May/June 2002

9 The complex number $1 + \mathrm{i}\sqrt{3}$ is denoted by u.

(i) Express u in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. Hence, or otherwise, find the modulus and argument of u^2 and u^3. [5]

(ii) Show that u is a root of the equation $z^2 - 2z + 4 = 0$, and state the other root of this equation. [2]

(iii) Sketch an Argand diagram showing the points representing the complex numbers i and u. Shade the region whose points represent every complex number z satisfying both the inequalities

$$|z - \mathrm{i}| \leqslant 1 \quad \text{and} \quad \arg z \geqslant \arg u.$$ [4]

Oct/Nov 2002

8 **(a)** Find the two square roots of the complex number $-3 + 4\mathrm{i}$, giving your answers in the form $x + \mathrm{i}y$, where x and y are real. [5]

(b) The complex number z is given by

$$z = \frac{-1 + 3\mathrm{i}}{2 + \mathrm{i}}.$$

(i) Express z in the form $x + \mathrm{i}y$, where x and y are real. [2]

(ii) Show on a sketch of an Argand diagram, with origin O, the points A, B and C representing the complex numbers $-1 + 3\mathrm{i}$, $2 + \mathrm{i}$ and z respectively. [1]

(iii) State an equation relating the lengths OA, OB and OC. [1]

May/June 2003

5 The complex number $2\mathrm{i}$ is denoted by u. The complex number with modulus 1 and argument $\frac{2}{3}\pi$ is denoted by w.

(i) Find in the form $x + \mathrm{i}y$, where x and y are real, the complex numbers w, uw and $\frac{u}{w}$. [4]

(ii) Sketch an Argand diagram showing the points U, A and B representing the complex numbers u, uw and $\frac{u}{w}$ respectively. [2]

(iii) Prove that triangle UAB is equilateral. [2]

Oct/Nov 2003

7 The complex number u is given by $u = \frac{7 + 4\mathrm{i}}{3 - 2\mathrm{i}}$.

(i) Express u in the form $x + \mathrm{i}y$, where x and y are real. [3]

(ii) Sketch an Argand diagram showing the point representing the complex number u. Show on the same diagram the locus of the complex number z such that $|z - u| = 2$. [3]

(iii) Find the greatest value of $\arg z$ for points on this locus. [3]

May/June 2004

8 **(i)** Find the roots of the equation $z^2 - z + 1 = 0$, giving your answers in the form $x + \mathrm{i}y$, where x and y are real. [2]

(ii) Obtain the modulus and argument of each root. [3]

(iii) Show that each root also satisfies the equation $z^3 = -1$. [2]

Oct/June 2004

6 The complex numbers $1 + 3\mathrm{i}$ and $4 + 2\mathrm{i}$ are denoted by u and v respectively.

(i) Find, in the form $x + \mathrm{i}y$, where x and y are real, the complex numbers $u - v$ and $\frac{u}{v}$. [3]

(ii) State the argument of $\frac{u}{v}$. [1]

In an Argand diagram, with origin O, the points A, B and C represent the numbers u, v and $u - v$ respectively.

(iii) State fully the geometrical relationship between OC and BA. [2]

(iv) Prove that angle $AOB = \frac{1}{4}\pi$ radians. [2]

May/June 2005

3 **(i)** Solve the equation $z^2 - 2\mathrm{i}z - 5 = 0$, giving your answers in the form $x + \mathrm{i}y$ where x and y are real. [3]

(ii) Find the modulus and argument of each root. [3]

(iii) Sketch an Argand diagram showing the points representing the roots. [1]

Oct/Nov 2005

7 The equation $2x^3 + x^2 + 25 = 0$ has one real root and two complex roots.

(i) Verify that $1 + 2\mathrm{i}$ is one of the complex roots. [3]

(ii) Write down the other complex root of the equation. [1]

(iii) Sketch an Argand diagram showing the point representing the complex number $1 + 2\mathrm{i}$. Show on the same diagram the set of points representing the complex numbers z which satisfy

$$|z| = |z - 1 - 2\mathrm{i}|.$$ [4]

May/June 2006

7 The complex number $2 + \mathrm{i}$ is denoted by u. Its complex conjugate is denoted by u^*.

(i) Show, on a sketch of an Argand diagram with origin O, the points A, B and C representing the complex numbers u, u^* and $u + u^*$ respectively. Describe in geometrical terms the relationship between the four points O, A, B and C. [4]

(ii) Express $\dfrac{u}{u^*}$ in the form $x + \mathrm{i}y$, where x and y are real. [3]

(iii) By considering the argument of $\dfrac{u}{u^*}$, or otherwise, prove that

$$\tan^{-1}\left(\tfrac{4}{3}\right) = 2\tan^{-1}\left(\tfrac{1}{2}\right).$$ [2]

Oct/Nov 2006

9 The complex number u is given by

$$u = \frac{3 + \mathrm{i}}{2 - \mathrm{i}}.$$

(i) Express u in the form $x + \mathrm{i}y$, where x and y are real. [3]

(ii) Find the modulus and argument of u. [2]

(iii) Sketch an Argand diagram showing the point representing the complex number u. Show on the same diagram the locus of the point representing the complex number z such that $|z - u| = 1$. [3]

(iv) Using your diagram, calculate the least value of $|z|$ for points on this locus. [2]

May/June 2007

8 The complex number $\dfrac{2}{-1 + \mathrm{i}}$ is denoted by u.

(i) Find the modulus and argument of u and u^2. [6]

(ii) Sketch an Argand diagram showing the points representing the complex numbers u and u^2. Shade the region whose points represent the complex numbers z which satisfy both the inequalities $|z| < 2$ and $|z - u^2| < |z - u|$. [4]

Oct/Nov 2007

8 **(a)** The complex number z is given by $z = \dfrac{4 - 3\mathrm{i}}{1 - 2\mathrm{i}}$.

(i) Express z in the form $x + \mathrm{i}y$, where x and y are real. [2]

(ii) Find the modulus and argument of z. [2]

(b) Find the two square roots of the complex number $5 - 12\mathrm{i}$, giving your answers in the form $x + \mathrm{i}y$, where x and y are real. [6]

May/June 2008

5 The variable complex number z is given by

$$z = 2\cos\theta + \mathrm{i}(1 - 2\sin\theta),$$

where θ takes all values in the interval $-\pi < \theta \leqslant \pi$.

(i) Show that $|z - \mathrm{i}| = 2$, for all values of θ. Hence sketch, in an Argand diagram, the locus of the point representing z. [3]

(ii) Prove that the real part of $\dfrac{1}{z + 2 - \mathrm{i}}$ is constant for $-\pi < \theta < \pi$. [4]

Oct/Nov 2008

10 The complex number w is given by $w = -\dfrac{1}{2} + \mathrm{i}\dfrac{\sqrt{3}}{2}$.

(i) Find the modulus and argument of w. [2]

(ii) The complex number z has modulus R and argument θ, where $-\frac{1}{3}\pi < \theta < \frac{1}{3}\pi$. State the modulus and argument of wz and the modulus and argument of $\dfrac{z}{w}$. [4]

(iii) Hence explain why, in an Argand diagram, the points representing z, wz and $\dfrac{z}{w}$ are the vertices of an equilateral triangle. [2]

(iv) In an Argand diagram, the vertices of an equilateral triangle lie on a circle with centre at the origin. One of the vertices represents the complex number $4 + 2\mathrm{i}$. Find the complex numbers represented by the other two vertices. Give your answers in the form $x + \mathrm{i}y$, where x and y are real and exact. [4]

May/June 2009

7 **(i)** Solve the equation $z^2 + (2\sqrt{3})\mathrm{i}z - 4 = 0$, giving your answers in the form $x + \mathrm{i}y$, where x and y are real. [3]

(ii) Sketch an Argand diagram showing the points representing the roots. [1]

(iii) Find the modulus and argument of each root. [3]

(iv) Show that the origin and the points representing the roots are the vertices of an equilateral triangle. [1]

May/June 2010/33

8 **(a)** The equation $2x^3 - x^2 + 2x + 12 = 0$ has one real root and two complex roots. Showing your working, verify that $1 + \mathrm{i}\sqrt{3}$ is one of the complex roots. State the other complex root. [4]

(b) On a sketch of an Argand diagram, show the point representing the complex number $1 + \mathrm{i}\sqrt{3}$. On the same diagram, shade the region whose points represent the complex numbers z which satisfy both the inequalities $|z - 1 - \mathrm{i}\sqrt{3}| \leqslant 1$ and $\arg z \leqslant \frac{1}{3}\pi$. [5]

Oct/Nov 2009/31

7 The complex number $-2 + \mathrm{i}$ is denoted by u.

(i) Given that u is a root of the equation $x^3 - 11x - k = 0$, where k is real, find the value of k. [3]

(ii) Write down the other complex root of this equation. [1]

(iii) Find the modulus and argument of u. [2]

(iv) Sketch an Argand diagram showing the point representing u. Shade the region whose points represent the complex numbers z satisfying both the inequalities

$$|z| < |z - 2| \quad \text{and} \quad 0 < \arg(z - u) < \tfrac{1}{4}\pi.$$ [4]

Oct/Nov 2009/32

7 The complex numbers $-2 + \mathrm{i}$ and $3 + \mathrm{i}$ are denoted by u and v respectively.

(i) Find, in the form $x + \mathrm{i}y$, the complex numbers

(a) $u + v$, [1]

(b) $\dfrac{u}{v}$, showing all your working. [3]

(ii) State the argument of $\dfrac{u}{v}$. [1]

In an Argand diagram with origin O, the points A, B and C represent the complex numbers u, v and $u + v$ respectively.

(iii) Prove that angle $AOB = \frac{3}{4}\pi$. [2]

(iv) State fully the geometrical relationship between the line segments OA and BC. [2]

May/June 2010/31

7 The complex number $2 + 2\mathrm{i}$ is denoted by u.

(i) Find the modulus and argument of u. [2]

(ii) Sketch an Argand diagram showing the points representing the complex numbers 1, i and u. Shade the region whose points represent the complex numbers z which satisfy both the inequalities $|z - 1| \leqslant |z - \mathrm{i}|$ and $|z - u| \leqslant 1$. [4]

(iii) Using your diagram, calculate the value of $|z|$ for the point in this region for which $\arg z$ is least. [3]

May/June 2010/32

8 The variable complex number z is given by

$$z = 1 + \cos 2\theta + \mathrm{i}\sin 2\theta,$$

where θ takes all values in the interval $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$.

(i) Show that the modulus of z is $2\cos\theta$ and the argument of z is θ. [6]

(ii) Prove that the real part of $\dfrac{1}{z}$ is constant. [3]