Integration – 2021 A2 Nov P3 Math

1. Nov/2021/Paper_9709/31/No.4

Using the substitution $u = \sqrt{x}$, find the exact value of

$$\int_3^\infty \frac{1}{(x+1)\sqrt{x}}\,dx. \qquad [6]$$

$u = x^{1/2} \Rightarrow \frac{du}{dx} = \frac{1}{2}x^{-1/2}$

$\Rightarrow du = \frac{1}{2}x^{-1/2}\,dx \Rightarrow \left\{du = \frac{1}{2}\left(\frac{1}{\sqrt{x}}\right)dx\right\}$

Now using integration by Substitution method.

$\Rightarrow \int_3^\infty \frac{1}{(u^2+1)\sqrt{x}} \cdot 2\sqrt{x}\,du = \int_0^\infty \frac{1}{(u^2+1)\cancel{\sqrt{x}}} \cdot 2\cancel{\sqrt{x}}\,du$

$= \int_0^\infty \frac{2}{u^2+1}\,du = 2\int_0^\infty \frac{1}{u^2+1}\,du$

$= 2\left[\tan^{-1} u\right] \Rightarrow 2\left[\tan^{-1}\sqrt{x}\right]_3^\infty$

$\Rightarrow \int_3^\infty \frac{1}{(x+1)\sqrt{x}}\,dx = 2\left[\tan^{-1}\sqrt{\infty} - \tan^{-1}\sqrt{3}\right]$

$= 2\left[\tan^{-1}\infty - \tan^{-1}\sqrt{3}\right]$

$= 2\left[\frac{\pi}{2} - \frac{\pi}{3}\right] = 2\left(\frac{\pi}{6}\right) = \frac{\cancel{2}\pi}{\cancel{6}\,3}$

$= \frac{\pi}{3}$

2. Nov/2021/Paper_9709/32/No.6(b)

(b) Hence show that $\int_0^{\frac{1}{4}\pi} \sin 3x \cos 2x \, dx = \frac{1}{5}(3 - \sqrt{2})$. [3]

Recall $\sin 3x \cos 2x = \frac{1}{2}(\sin 5x + \sin x)$

$\Rightarrow I = \int_0^{\pi/4} \sin 3x \cos 2x \, dx = \int_0^{\pi/4} \frac{1}{2}[\sin 5x + \sin x] \, dx$

$= \frac{1}{2}\int_0^{\pi/4} \sin 5x \, dx + \frac{1}{2}\int_0^{\pi/4} \sin x \, dx$

$\Rightarrow \frac{1}{2}\int_0^{\pi/4} \sin 5x \, dx = \frac{1}{2}\left[-\frac{1}{5}\cos 5x\right]_0^{\pi/4}$

$= -\frac{1}{10}\left[\cos 5x\right]_0^{\pi/4} = -\frac{1}{10}\left\{\cos \frac{5}{4}\pi - \cos 0\right\}$

$= -\frac{1}{10}\left\{-\frac{\sqrt{2}}{2} - 1\right\} = \frac{\sqrt{2}}{20} + \frac{1}{10}$

and $\frac{1}{2}\int_0^{\pi/4} \sin x \, dx = \left[-\frac{1}{2}\cos x\right]_0^{\pi/4}$

$= -\frac{1}{2}\left\{\cos \frac{\pi}{4} - \cos 0\right\} = -\frac{1}{2}\left\{\frac{\sqrt{2}}{2} - 1\right\}$

$= -\frac{\sqrt{2}}{4} + \frac{1}{2}$

$\Rightarrow \int_0^{\pi/4} \sin 3x \cos 2x \, dx = \frac{\sqrt{2}}{20} + \frac{1}{10} - \frac{\sqrt{2}}{4} + \frac{1}{2}$

$= -\frac{1}{5}\sqrt{2} + \frac{3}{5} = \frac{3}{5} - \frac{1}{5}\sqrt{2}$

$= \frac{3}{5} - \frac{1}{5}\sqrt{2} = \frac{1}{5}(3 - \sqrt{2})$ Shown.

3. Nov/2021/Paper_9709/33/No.4

Find the exact value of $\int_{\frac{1}{3}\pi}^{\pi} x \sin \frac{1}{2}x \, dx$. [5]

let $I = \int x \sin x/_2 \, dx$

$I = \int x \sin \frac{1}{2}x \, dx$; using integration by parts

$\Rightarrow I = \int x \sin \frac{1}{2}x \, dx = uv - \int v \frac{du}{dx} dx$

let $u = x$
$\Rightarrow \frac{du}{dx} = 1$

and $\frac{dv}{dx} = \sin \frac{1}{2}x \Rightarrow v = -2\cos \frac{1}{2}x$

$v = \int \sin \frac{1}{2}x \, dx$

$I = (x)\left(-2\cos \frac{1}{2}x\right) - \int -2\cos \frac{1}{2}x \, dx$

$= -2x \cos \frac{1}{2}x + 2\int \cos \frac{1}{2}x \, dx$

$= -2x \cos \frac{1}{2}x + 4 \sin \frac{1}{2}x + k$

$\Rightarrow \int_{\frac{1}{3}\pi}^{\pi} x \sin \frac{1}{2}x = \left[-2x \cos \frac{1}{2}x + 4 \sin \frac{1}{2}x\right]_{\pi/3}^{\pi}$

$= \left[-2x \cos \frac{1}{2}x + 4 \sin \frac{1}{2}x\right]_{\pi/3}^{\pi}$

$= \left[-2(\pi)\cos \frac{\pi}{2} + 4 \sin \pi/_2\right] - \left[-2(\pi/_3)\cos \frac{\pi}{6} + 4 \sin \pi/_6\right]$

$= \left[-2\pi(0) + 4(1)\right] - \left[-2/_3\pi\left(\frac{\sqrt{3}}{2}\right) + 4\left(\frac{1}{2}\right)\right]$

$= [0 + 4] - \left[-\frac{\sqrt{3}}{3}\pi + 2\right] = 4 - -\frac{\sqrt{3}}{3}\pi - 2$

$= 2 + \frac{\sqrt{3}}{3}\pi$

4. Nov/2021/Paper_9709/33/No.9

Let $f(x) = \dfrac{1}{(9-x)\sqrt{x}}$.

(a) Find the x-coordinate of the stationary point of the curve with equation $y = f(x)$. [4]

$$f(x) = \frac{1}{(9-x)\sqrt{x}} = \frac{1}{(9-x)x^{\frac{1}{2}}}$$

$$f(x) = \frac{1}{9x^{\frac{1}{2}} - x^{\frac{3}{2}}}$$

using quotient rule

$$f'(x) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

for stationarity $f'(x) = 0$

$u = 1 \Rightarrow \frac{du}{dx} = 0$

$v = 9x^{\frac{1}{2}} - x^{\frac{3}{2}}$

$\frac{dv}{dx} = 4.5x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}}$

$$\frac{(9x^{\frac{1}{2}} - x^{\frac{3}{2}})0 - \left[1\left(\frac{9}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}}\right)\right]}{(9x^{\frac{1}{2}} - x^{\frac{3}{2}})^2} = 0$$

$$\Rightarrow 0 - \left[\frac{9}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}}\right] = 0$$

$$-\frac{4.5}{\sqrt{x}} + 1.5\sqrt{x} = 0$$

$$-\frac{4.5}{\cancel{\sqrt{x}}}(\cancel{\sqrt{x}}) + 1.5(\sqrt{x})^2 = 0\sqrt{x}$$

$$-4.5 + 1.5x = 0$$

$$\Rightarrow \frac{1.5x}{1.5} = \frac{4.5}{1.5}$$

$$x = \frac{4.5}{1.5} = 3$$

$$\Rightarrow x = 3$$

(b) Using the substitution $u = \sqrt{x}$, show that $\int_0^4 f(x)\,dx = \frac{1}{3}\ln 5$. [6]

$u = \sqrt{x} \Rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow dx = 2\sqrt{x}\,du$

When $x = 0 \Rightarrow u = \sqrt{0} = 0$ and when $x = 4 \Rightarrow u = \sqrt{4} = 2$

$\Rightarrow \int_0^4 \frac{1}{(9-x)\sqrt{x}}\,dx = \int_0^2 \frac{2\cancel{\sqrt{x}}\cdot du}{(9-u^2)\cancel{\sqrt{x}}} = 2\int_0^2 \frac{1}{9-u^2}\,du$

$= 2\int_0^2 \frac{1}{(3+u)(3-u)} \Rightarrow \frac{1}{(3+u)(3-u)} = \frac{A}{3+u} + \frac{B}{3-u}$

$\Rightarrow 1 = A(3-u) + B(3+u)$; when $u = -3$

$\Rightarrow 1 = A(3--3) + 0 \Rightarrow 1 = 6A \Rightarrow \boxed{A = 1/6}$

when $u = 3 \Rightarrow$

$1 = 0 + B(3+3) \Rightarrow 1 = 6B \Rightarrow \boxed{B = \frac{1}{6}}$

$\Rightarrow 2\int_0^2 \frac{1}{(3+u)(3-u)}\,du = 2\int_0^2 \left(\frac{1/6}{3+u} + \frac{1/6}{3-u}\right)du$

$= \frac{1}{3}\int_0^2 \frac{1}{3+u}\,du + \frac{1}{3}\int_0^2 \frac{1}{3-u}\,du$

$= \frac{1}{3}\left\{\left[\ln(3+u) - \ln(3-u)\right]_0^2\right\}$

$= \frac{1}{3}\left[\ln\left(\frac{3+u}{3-u}\right)\right]_0^2 = \frac{1}{3}\left\{\ln\left(\frac{5}{1}\right) - \ln\frac{3}{3}\right\}$

$= \frac{1}{3}\{\ln 5 - \ln 1\} = \frac{1}{3}[\ln 5 - 0]$

$= 1/3 \ln 5$ shown.