Continuous Random Variables – 2022 A2 June Math

1. March/2022/Paper_9709/62/No.6

In a game a ball is rolled down a slope and along a track until it stops. The distance, in metres, travelled by the ball is modelled by the random variable X with probability density function

$$f(x) = \begin{cases} -k(x-1)(x-3) & 1 \leqslant x \leqslant 3, \\ 0 & \text{otherwise,} \end{cases}$$

where k is a constant.

(a) Without calculation, explain why $E(X) = 2$. [1]

Quadratic with roots $x=1$ and $x=3$ so symmetrical about $x = \frac{1+3}{2} = 2$.

(b) Show that $k = \frac{3}{4}$. [3]

$\Rightarrow \int_1^3 -k(x-1)(x-3)\,dx = 1.0$ since $f(x)$ is a p.d.f

$\Rightarrow -k\int_1^3 (x^2 - 4x + 3)\,dx = 1$

$\Rightarrow -k\left[\frac{x^3}{3} - \frac{4x^2}{2} + 3x\right]_1^3 = 1$

$\Rightarrow \left[\frac{x^3}{3} - 2x^2 + 3x\right]_1^3 = \frac{1}{-k}$

$\left[\frac{(3)^3}{3} - 2(3)^2 + 3(3)\right] - \left[\frac{1}{3} - 2 + 3\right] = \frac{1}{-k}$

$[9 - 18 + 9] - \left[\frac{1}{3} - 2 + 3\right] = \frac{1}{-k}$

$(-9+9) - \left(\frac{4}{3}\right) = -\frac{1}{k}$

$-\frac{4}{3} = -\frac{1}{k}$

$\frac{-4k}{-4} = \frac{-3}{-4}$

$k = \frac{3}{4}$

Shown.

(c) Find Var(X). [3]

$Var(x) = -\frac{3}{4}\int_1^3 x^2 \cdot f(x)\,dx - [E(x)]^2$

$= \frac{3}{4}\int_1^3 (x^4 - 4x^3 + 3x^2)\,dx - [E(x)]^2$

Recall $Var(x) = E(x^2) - [E(x)]^2$

$= -\frac{3}{4}\left\{\left[\frac{x^5}{5} - x^4 - x^3\right]_1^3 - 2^2\right\}$

$= -\frac{3}{4}\left[\left(\frac{243}{5} - 81 + 27\right) - \left(\frac{1}{5} - 1 + 1\right)\right.$

$- 4$

$= -\frac{3}{4}\left[-\frac{27}{5} - \frac{1}{5}\right] - 4$

$= -\frac{3}{4}\left(-\frac{28}{5}\right) - 4$

$= 4.2 - 4$

$= 0.2$

One turn consists of rolling the ball 3 times and noting the largest value of X obtained. If this largest value is greater than 2.5, the player scores a point.

(d) Find the probability that on a particular turn the player scores a point. [4]

$= -\frac{3}{4}\int_{2.5}^3 (x^2 - 4x + 3)\,dx$

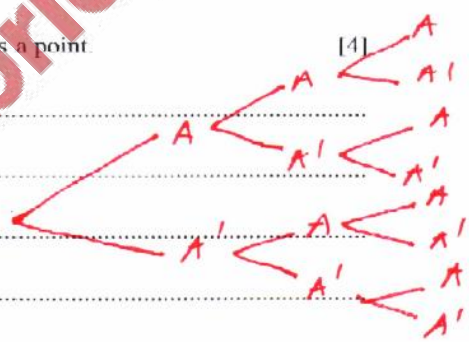
$= -\frac{3}{4}\left[\frac{x^3}{3} - 2x^2 + \cancel{4}3x\right]_{2.5}^3$

$= -\frac{3}{4}\left[(\overset{0}{9 - 18 + 9}) - \left(\frac{125}{24} - \frac{25}{2} + 7.5\right)\right]$

$= -\frac{3}{4}\left(-\frac{5}{24}\right)$

$= \frac{5}{32}$

$P(X > 2.5) = P(A)$



$P(\text{score a point})$

$= 1 - \left(1 - \frac{5}{32}\right)^3$

$= 1 - 0.60068$

$= 0.399$

$= 0.399.$