1. March/2022/Paper_9709/32/No.9

The variables x and y satisfy the differential equation

$$(x+1)(3x+1)\frac{dy}{dx} = y,$$

and it is given that $y = 1$ when $x = 1$.

Solve the differential equation and find the exact value of y when $x = 3$, giving your answer in a simplified form. [9]

$(x+1)(3x+1)\dfrac{dy}{dx} = y$ → it's a differential Equation with separable variables.

Boundary Condition ⇒ $y = 1$ when $x = 1$

$\dfrac{1}{y}\, dy = \dfrac{1}{(x+1)(3x+1)}\, dx$

$\Rightarrow \dfrac{1}{(x+1)(3x+1)} = \dfrac{A}{(x+1)} + \dfrac{B}{(3x+1)}$;

$\Rightarrow 1 = A(3x+1) + B(x+1) \Rightarrow$ when $x = -\frac{1}{3}$

$\Rightarrow 1 = 0 + B(-\frac{1}{3}+1) \Rightarrow 1 = \frac{2}{3}B \Rightarrow \boxed{B = \frac{3}{2}}$

when $x = -1$

$\Rightarrow 1 = A(-3+1) + 0 \Rightarrow \dfrac{1}{-2} = \dfrac{-2}{-2}A \Rightarrow \boxed{A = -\frac{1}{2}}$

$\Rightarrow \dfrac{1}{y}\, dy = \dfrac{-\frac{1}{2}}{x+1} + \dfrac{\frac{3}{2}}{(3x+1)}\, dx$

$\int \dfrac{1}{y}\, dy = \int \dfrac{-\frac{1}{2}}{(x+1)}\, dx + \int \dfrac{\frac{3}{2}}{(3x+1)}\, dx$

$\Rightarrow \ln y = -\dfrac{1}{2}\int \dfrac{1}{x+1}\, dx + \dfrac{3}{2}\int \dfrac{1}{(3x+1)}\, dx$

$\Rightarrow \ln y = -\dfrac{1}{2}\ln(x+1) + \left(\dfrac{3}{2}\right)\left(\dfrac{1}{3}\right)\ln(3x+1) + C$

$\Rightarrow \ln y = -\frac{1}{2}\ln(x+1) + \dfrac{1}{2}\ln(3x+1) + C$

But we know $y = 1$ when $x = 1$

$$\ln 1 = -\frac{1}{2}\ln 2 + \frac{1}{2}\ln 4 + c$$

$$0 = -\frac{1}{2}\left[\ln 2 - \ln 4\right] + c$$

$$\Rightarrow 0 = -\frac{1}{2}\left[\ln\left(\frac{2}{4}\right) + c\right]$$

$$\Rightarrow 0 = -\frac{1}{2}\ln\frac{1}{2} + c$$

$$0 = -\frac{1}{2}\ln 2^{-1} + c$$

$$0 = -\frac{1}{2}(-1)\ln 2 + c$$

$$0 = \frac{1}{2}\ln 2 + c$$

$$c = -\frac{1}{2}\ln 2$$

$\Rightarrow$ particular solution will be

$$\ln y = -\frac{1}{2}\ln(x+1) + \frac{1}{2}\ln(3x+1) - \frac{1}{2}\ln 2$$

$$\Rightarrow \ln y = \frac{1}{2}\ln(3x+1) - \frac{1}{2}\ln(x+1) - \frac{1}{2}\ln 2$$

$$\ln y = \frac{1}{2}\left[\ln(3x+1) - \left(\ln(x+1) + \ln 2\right)\right]$$

$$\ln y = \frac{1}{2}\left[\ln(3x+1) - \ln(2x+2)\right]$$

$$\ln y = \frac{1}{2}\ln\left(\frac{3x+1}{2x+2}\right)$$

$$\ln y = \ln\sqrt{\frac{3x+1}{2x+2}}$$

$$\Rightarrow y = \sqrt{\frac{3x+1}{2x+2}}$$

When $x = 3$

$$y = \sqrt{\frac{3(3)+1}{2(3)+2}}$$

$$y = \sqrt{\frac{10}{8}}$$

$$y = \frac{\pm\sqrt{5}}{2}$$

$$\Rightarrow y = \frac{1}{2}\sqrt{5}$$

2. June/2022/Paper_9709/31/No.4

The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = \frac{xy}{1+x^2},$$

and $y = 2$ when $x = 0$.

Solve the differential equation, obtaining a simplified expression for y in terms of x. [7]

$\frac{dy}{dx} = \frac{xy}{1+x^2}$ → differential Equation with separable variables

$\Rightarrow \frac{dy}{y} = \frac{x}{1+x^2}\,dx \Rightarrow \int \frac{dy}{y} = \int \frac{x}{1+x^2}\,dx$

$\ln y = \int \frac{x}{1+x^2}\,dx$ so; $\int \frac{x}{1+x^2}\,dx$;

now using integration by substitution method let $u = 1+x^2$

$\Rightarrow du = 2x\,dx \Rightarrow \boxed{dx = \frac{du}{2x}}$

$\Rightarrow \ln y = \int \frac{x}{1+x^2}\,dx$

$\Rightarrow \ln y = \int \frac{x}{u} \cdot \frac{du}{2x} + K$

$\Rightarrow \ln y = \int \frac{1}{u} \cdot \frac{1}{2}\,du + K$

$\ln y = \frac{1}{2} \ln u + K$

$\ln y = \frac{1}{2} \ln(1+x^2) + K$

Recall $y = 2$ when $x = 0$

$\ln y = \frac{1}{2}\ln(1+0^2) + k \Rightarrow$

$\ln y = \frac{1}{2}\ln 1 + k$ {But $\ln 1 = 0$}

$\ln y = 0 + k \Rightarrow$ {$k = \ln 2$}

particular solution will be

$\ln y = \frac{1}{2}\ln(1+x^2) + \ln 2$

$\ln y = \ln\left(\sqrt{1+x^2}\right) + \ln 2$

$\ln y = \ln\left(\sqrt{1+x^2}\right) + \ln 2$

$\Rightarrow$ Recall $\ln(AB) = \ln A + \ln B$

$\ln y = \ln\left[\left(\sqrt{1+x^2}\right)(2)\right]$

$\Rightarrow \ln y = \ln 2\left(\sqrt{1+x^2}\right)$

$y = 2\sqrt{1+x^2}$

3. June/2022/Paper_9709/32/No.6

The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = xe^{y-x},$$

and $y = 0$ when $x = 0$.

(a) Solve the differential equation, obtaining an expression for y in terms of x. [7]

Boundary condition $\Rightarrow$ $y = 0$ when $x = 0$

$\Rightarrow \frac{dy}{dx} = xe^{y-x} \Rightarrow \frac{dy}{dx} = \frac{xe^y}{e^x} \Rightarrow$

$\frac{dy}{dx} = xe^y e^{-x} \Rightarrow \frac{dy}{e^y} = \frac{x\cancel{e^y}e^{-x}dx}{\cancel{e^y}}$

$\Rightarrow \frac{1}{e^y}dy = xe^{-x}dx$

$\Rightarrow \int e^{-y}dy = \int xe^{-x}dx$ Now using

integration by parts $\Rightarrow \int xe^{-x}dx = uv - \int v\frac{du}{dx}dx$

$\Rightarrow$ let $\boxed{u = x \Rightarrow \frac{du}{dx} = 1}$ and $\frac{dv}{dx} = e^{-x} \Rightarrow v = \int e^{-x}dx$

$\Rightarrow v = -e^{-x}$

$\Rightarrow \int xe^{-x}dx = (x)(-e^{-x}) - \int -e^{-x}.(1)dx$

$= -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x}$

$\Rightarrow \int e^{-y}dy = \int xe^{-x}dx$

$-e^{-y} = -xe^{-x} - e^{-x} + C \Rightarrow$ General solution

But $y = 0$ when $x = 0$

$\Rightarrow -e^{-(0)} = 0 - e^{-0} + C$

$-1 = 0 - 1 + C$

$-1 = -1 + C$

$C = -1 + 1$

$C = 0$

$\therefore$ The particular solution will be

$-e^{-y} = -xe^{-x} - e^{-x} + 0$

$e^{-y} = xe^{-x} + e^{-x}$

$e^{-y} = e^{-x}(x+1)$; Now introduce ln on both sides i.e L.H.S & R.H.S

$\Rightarrow \ln e^{-y} = \ln\left[e^{-x}(x+1)\right]$; but recall $\ln a^b = b\ln a$

$\Rightarrow -y\ln e = \ln\left[(x+1)e^{-x}\right]$; But $\ln e = 1$

$\Rightarrow -y = \ln\left[(x+1)e^{-x}\right]$

$\Rightarrow y = -\ln\left[(x+1)e^{-x}\right]$.

(b) Find the value of y when $x = 1$, giving your answer in the form $a - \ln b$, where a and b are integers. [1]

$y = -\ln\left[(x+1)e^{-x}\right]$

$y = -\left\{\ln[x+1] + \ln e^{-x}\right\}$

$y = -\left\{\ln(x+1) - x\right\} = x - \ln(x+1)$

But when $x = 1$

$\Rightarrow y = 1 - \ln(1+1)$

$y = 1 - \ln 2$ where $\boxed{a=1}$ and $\boxed{b=2}$

4. June/2022/Paper_9709/33/No.8

At time t days after the start of observations, the number of insects in a population is N. The variation in the number of insects is modelled by a differential equation of the form $\frac{dN}{dt} = kN^{\frac{3}{2}}\cos 0.02t$, where k is a constant and N is a continuous variable. It is given that when $t = 0$, $N = 100$.

(a) Solve the differential equation, obtaining a relation between N, k and t. [5]

$$\frac{dN}{dt} = kN^{3/2}\cos(0.02t)$$

$N = 100$ when $t = 0$

Recall $\int \cos ax\,dx = \frac{1}{a}\sin ax + c$

$$\Rightarrow \frac{dN}{N^{3/2}} = k\cos(0.02t)\,dt$$

$$\Rightarrow \int N^{-3/2}\,dN = \int k\cos(0.02t)\,dt$$

$$\frac{N^{-0.5}}{-0.5} = k\left[\frac{\sin(0.02t)}{0.02}\right] + c$$

$$-\frac{2}{\sqrt{N}} = k\left[50\sin(0.02t)\right] + c$$

$$-\frac{2}{\sqrt{N}} = k\left[50\sin(0.02t)\right] + c$$

Recall the boundary condition: $N = 100$ when $t = 0$

$$-\frac{2}{\sqrt{100}} = k\left[50\sin 0\right] + c$$

$$= 0 + c$$

$$-\frac{2}{\sqrt{100}} \quad c = -0.2$$

$$c = -\frac{1}{5}$$

$\therefore$ The particular solution.

$$-\frac{2}{\sqrt{N}} = k\left[50\sin(0.02t)\right] - \frac{1}{5}$$

The particular solution.

$$-\frac{2}{\sqrt{N}} = k\left[50\sin(0.02t)\right] - \frac{1}{5}$$

$$-\frac{2}{\sqrt{N}} = k\left[50\sin(0.02t)\right] - \frac{1}{5}$$

(b) Given also that $N = 625$ when $t = 50$, find the value of k. [2]

When $t=50$ then $N=625$

$\Rightarrow -\frac{2}{25} = 50k\sin 1 - \frac{1}{5}$

$\frac{3}{25} = 50k\sin 1$

$\Rightarrow k = \frac{3}{50 \times 25 \times \sin 1}$

$K = 0.00285$

(c) Obtain an expression for N in terms of t, and find the greatest value of N predicted by this model. [2]

$-\frac{2}{\sqrt{N}} = \left(0.1426\sin 0.02t - \frac{1}{5}\right)^2$

$\frac{4}{\sqrt{N}} = \left(0.1426\sin 0.02t - \frac{1}{5}\right)^2$

$N = \frac{4}{\left(0.1426\sin 0.02t - \frac{1}{5}\right)^2}$

Greatest N

$\sin 0.02t = 1$

Greatest N

$\Rightarrow N = \frac{4}{\left(0.1426 - \frac{1}{5}\right)^2}$

$N \simeq 1210$

$\Rightarrow N = 1210$