1. Nov/2022/Paper_9709_31/No.4

Solve the equation $\tan(x + 45°) = 2\cot x$ for $0° < x < 180°$. [5]

$$\boxed{\begin{aligned} \cot x &= \frac{1}{\tan x} \\ \tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \end{aligned}}$$

$\Rightarrow \tan(x + 45°) = 2\cot x$

$$\frac{\tan x + \tan 45°}{1 - \tan x \tan 45°} = \frac{2}{\tan x}$$

$$\frac{\tan x + 1}{1 - \tan x} = \frac{2}{\tan x}$$

By cross-multiplication:

$\tan x(\tan x + 1) = 2(1 - \tan x)$

$\tan^2 x + \tan x = 2 - 2\tan x$

$\Rightarrow \tan^2 x + \tan x + 2\tan x - 2 = 0$

$\tan^2 x + 3\tan x - 2 = 0$

Let $\tan x = y$

$\Rightarrow y^2 + 3y - 2 = 0$

Solving for y using the quadratic formular:

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = \frac{-3 \pm \sqrt{3^2 - 4(1)(-2)}}{2(1)}$$

$$y = \frac{-3 \pm \sqrt{17}}{2}$$

$$\Rightarrow \tan x = \frac{-3 \pm \sqrt{17}}{2}$$

When $\tan x = \dfrac{-3 + \sqrt{17}}{2}$

$$x = \tan^{-1}\left(\frac{-3 + \sqrt{17}}{2}\right)$$

$x = 29.3°$, $180 + 29.3° = 209.3°$

When $\tan x = \dfrac{-3 - \sqrt{17}}{2}$

$$x = \tan^{-1}\left(\frac{-3 - \sqrt{17}}{2}\right)$$

$x = -74.3°$, $180° - 74.3° = 105.7°$

In the interval $0° < x < 180°$

$x = 29.3°, 105.7°$

2. Nov/2022/Paper_9709_31/No.6

(a) Prove the identity $\cos 4\theta + 4\cos 2\theta + 3 \equiv 8\cos^4\theta$. [4]

Using the double angle formular $\cos 2\theta = 2\cos^2\theta - 1$

$\cos 4\theta + 4\cos 2\theta + 3$

$\cos 2(2\theta) + 4(2\cos^2\theta - 1) + 3$

Let $2\theta = x$

$\cos 2x + 4(2\cos^2\theta - 1) + 3$

$2\cos^2 x - 1 + 8\cos^2\theta - 4 + 3$

Substituting x with 2θ

$2\cos^2 2\theta - 1 + 8\cos^2\theta - 1$

$2(\cos 2\theta)^2 - 2 + 8\cos^2\theta$

$2[2\cos^2\theta - 1]^2 - 2 + 8\cos^2\theta$

$2[2\cos^2\theta(2\cos^2\theta - 1) - 1(2\cos^2\theta - 1)] - 2 + 8\cos^2\theta$

$2[4\cos^4\theta - 2\cos^2\theta - 2\cos^2\theta + 1] - 2 + 8\cos^2\theta$

$2[4\cos^4\theta - 4\cos^2\theta + 1] - 2 + 8\cos^2\theta$

$\Rightarrow 8\cos^4\theta - 8\cos^2\theta + 2 - 2 + 8\cos^2\theta$

$8\cos^4\theta + (8\cos^2\theta - 8\cos^2\theta) + (2-2)$

$8\cos^4\theta + 0 + 0$

$= 8\cos^4\theta$

$\therefore \cos 4\theta + 4\cos 2\theta + 3 \equiv 8\cos^4\theta$

(b) Hence solve the equation $\cos 4\theta + 4\cos 2\theta = 4$ for $0° \leqslant \theta \leqslant 180°$. [3]

From (a) $\cos 4\theta + 4\cos 2\theta + 3 = 8\cos^4\theta$

$\Rightarrow \cos 4\theta + 4\cos 2\theta = 8\cos^4\theta - 3$

Comparing with $\cos 4\theta + 4\cos 2\theta = 4$

$\Rightarrow 8\cos^4\theta - 3 = 4$

$8\cos^4\theta = 4 + 3$

$8\cos^4\theta = 7$

$\cos^4\theta = \frac{7}{8}$

$\Rightarrow \cos\theta = \sqrt[4]{\frac{7}{8}}$

$\cos\theta = \pm 0.967168$

$\cos\theta = -0.967168$, $\cos\theta = 0.967168$

$\theta = \cos^{-1}(-0.967168)$, $\theta = \cos^{-1}(0.967168)$

$\theta = 165.3°$, $\theta = 14.7°$

In the interval $0° \leq \theta \leq 180°$

$\theta = 14.7°, 165.3°$

3. Nov/2022/Paper_9709_32/No.4

(a) Express $4\cos x - \sin x$ in the form $R\cos(x + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. State the exact value of R and give α correct to 2 decimal places. [3]

$4\cos x - \sin x = R\cos(x + \alpha)$

$= R\cos x\cos\alpha - R\sin x\sin\alpha$

$\Rightarrow R\cos\alpha = 4$

$R\sin\alpha = 1$

$(R\cos\alpha)^2 + (R\sin\alpha)^2 = 4^2 + 1^2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 17$

$R^2(\cos^2\alpha + \sin^2\alpha) = 17$

But $\cos^2\alpha + \sin^2\alpha = 1$

$\Rightarrow R^2 = 17$

$\therefore R = \sqrt{17}$

$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{1}{4}$

$\alpha = \tan^{-1}(1/4)$

$\alpha = 14.04°$ (2d.p)

$\therefore 4\cos x - \sin x = \sqrt{17}\cos(x + 14.04°)$

(b) Hence solve the equation $4\cos 2x - \sin 2x = 3$ for $0° < x < 180°$. [5]

From (a) $4\cos x - \sin x = \sqrt{17}\cos(x + 14.04°)$

$\Rightarrow 4\cos 2x - \sin 2x = \sqrt{17}\cos(2x + 14.04°)$

$= 3$

$\sqrt{17}\cos(2x + 14.04°) = 3$

$\cos(2x + 14.04°) = \frac{3}{\sqrt{17}}$

$\Rightarrow 2x + 14.04° = \cos^{-1}\left(\frac{3}{\sqrt{17}}\right)$

$2x + 14.04° = 43.3°, 360 - 43.3°$

$2x + 14.04° = 43.3°, 316.7°$

$2x = 43.3° - 14.04°, 316.7° - 14.04°$

$2x = 29.26, 302.66$

$x = \frac{29.26}{2}, \frac{302.66}{2}$

$x = 14.6°, 151.3°$

$\therefore$ In the interval $0° < x < 180°$

$x = 14.6°, 151.3°$

4. Nov/2022/Paper_9709_33/No.7

(a) Show that the equation $\sqrt{5}\sec x + \tan x = 4$ can be expressed as $R\cos(x+\alpha) = \sqrt{5}$, where $R > 0$ and $0° < \alpha < 90°$. Give the exact value of R and the value of α correct to 2 decimal places. [4]

Recall that $a\cos x - b\sin x = R\cos(x+\alpha)$

$\sqrt{5}\sec x + \tan x = 4$, $\sec x = \frac{1}{\cos x}$

$\Rightarrow \frac{\sqrt{5}}{\cos x} + \frac{\sin x}{\cos x} = 4$

Multiplying by $\cos x$

$\left(\frac{\sqrt{5}}{\cos x} + \frac{\sin x}{\cos x} = 4\right)\cos x$

$\sqrt{5} + \sin x = 4\cos x$

$\Rightarrow 4\cos x - \sin x = \sqrt{5}$

$R\cos(x+\alpha) = R\cos x\cos\alpha - R\sin x\sin\alpha$

$\Rightarrow 4\cos x - \sin x = R\cos x\cos\alpha - R\sin x\sin\alpha = \sqrt{5}$

$\Rightarrow R\cos\alpha = 4$, $R\sin\alpha = 1$

$(R\cos\alpha)^2 + (R\sin\alpha)^2 = 4^2 + 1^2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 17$

$R^2(\cos^2\alpha + \sin^2\alpha) = 17$

But $\cos^2\alpha + \sin^2\alpha = 1$

$\Rightarrow R^2 = 17$

$\therefore R = \sqrt{17}$

$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{1}{4}$

$\Rightarrow \alpha = \tan^{-1}\left(\frac{1}{4}\right)$

$\alpha = 14.04°$ (2 d.p)

$\therefore R = \sqrt{17}$, $\alpha = 14.04°$

$\therefore \sqrt{17}\cos(x + 14.04°) = \sqrt{5}$

(b) Hence solve the equation $\sqrt{5}\sec 2x + \tan 2x = 4$, for $0° < x < 180°$. [4]

From (a) $\sqrt{5}\sec x + \tan x = 4 \equiv \sqrt{17}\cos(x + 14.04°) = \sqrt{5}$

$\Rightarrow \sqrt{5}\sec 2x + \tan 2x = 4 \equiv \sqrt{17}\cos(2x + 14.04°) = \sqrt{5}$

$\Rightarrow \cos(2x + 14.04°) = \dfrac{\sqrt{5}}{\sqrt{17}}$

$\Rightarrow 2x + 14.04° = \cos^{-1}\left(\dfrac{\sqrt{5}}{\sqrt{17}}\right)$

$2x + 14.04° = 57.16°, \quad 360° - 57.16° = 302.84°$

$2x + 14.04° = 57.16°, \quad 302.84°$

$\Rightarrow 2x = 57.16° - 14.04°, \quad 302.84° - 14.04°$

$2x = 43.12°, \quad 288.8°$

$x = \dfrac{43.12°}{2}, \quad \dfrac{288.8°}{2}$

$x = 21.6°, \quad 144.4°$

$\therefore$ In the interval $0° < x < 180°$,

$x = 21.6°, \quad 144.4°$