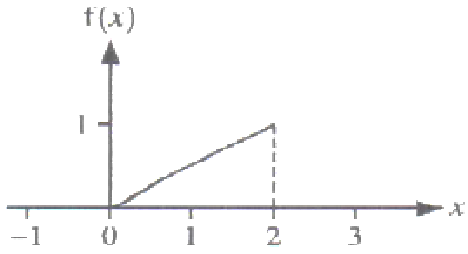
1. June/2023/Paper_9709/61/No.2

(a)



The graph of the function f is a straight line segment from (0, 0) to (2, 1).

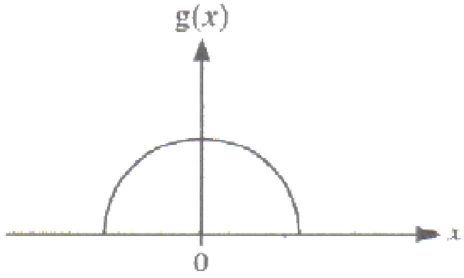
Show that f could be a probability density function. [2]

Area under the graph = Area of the triangle

$= \frac{1}{\cancel{2}} \times \cancel{2} \times 1 = 1$

and $f(x) \geq 0$ so f is a probability density function.

(b)



The graph of the function g is a semicircle, centre (0, 0), entirely above the x-axis.

Given that g is a probability density function, find the radius of the semicircle. [2]

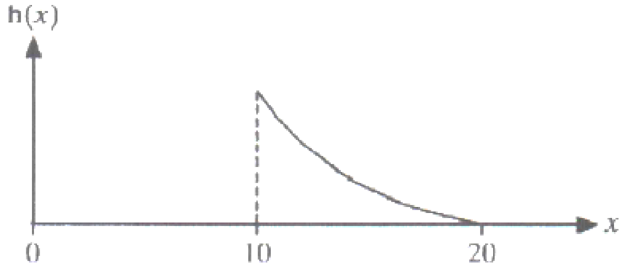
Area = 1

$\Rightarrow$ Area of the semicircle $= \frac{1}{2}\pi r^2 = 1$

$r^2 = \frac{2}{\pi}$

$r = \sqrt{\frac{2}{\pi}}$

(c)



The time, X minutes, taken by a large number of students to complete a test has probability density function h, as shown in the diagram.

(i) Without calculation, use the diagram to explain how you can tell that the median time is less than 15 minutes. [1]

The distribution of X is skewed to the right so the median time will be less than the mid-point of the interval which is 15.

It is now given that

$$h(x) = \begin{cases} \dfrac{40}{x^2} - \dfrac{1}{10} & 10 \leqslant x \leqslant 20, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) Find the mean time. [3]

$$\text{Mean} = \int_{10}^{20} x\,h(x)\,dx$$

$$= \int_{10}^{20} x\left(\frac{40}{x^2} - \frac{1}{10}\right) dx$$

$$= \int_{10}^{20} \left(\frac{40}{x} - \frac{1}{10}x\right) dx$$

$$= 40\int_{10}^{20} \frac{1}{x}\,dx - \frac{1}{10}\int_{10}^{20} x\,dx$$

$$= \left. 40\ln x - \frac{1}{10}\left(\frac{x^2}{2}\right)\right|_{10}^{20}$$

$$\left. 40\ln x - \frac{x^2}{20}\right|_{10}^{20}$$

$$\left(40\ln 20 - \frac{20^2}{20}\right) - \left(40\ln 10 - \frac{10^2}{20}\right)$$

$$= 40\ln 20 - 20 - 40\ln 10 + 5$$

$$= 40(\ln 20 - \ln 10) - 15$$

Using laws of logarithms $\ln\left(\frac{A}{B}\right) = \ln A - \ln B$

$$= 40\ln\left(\frac{20}{10}\right) - 15$$

$$= 40\ln 2 - 15$$

$$= 12.7 \quad (3\text{ s.f.})$$

2. June/2023/Paper_9709/63/No.1

A random variable X has probability density function f, where

$$f(x) = \begin{cases} \frac{3}{2}(1 - x^2) & 0 \leqslant x \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find E(X). [3]

$$E(x) = \int_0^1 x\, f(x)\, dx = \int_0^1 x\left(\frac{3}{2}(1-x^2)\right) dx$$

$$= \frac{3}{2}\int_0^1 (x - x^3)\, dx$$

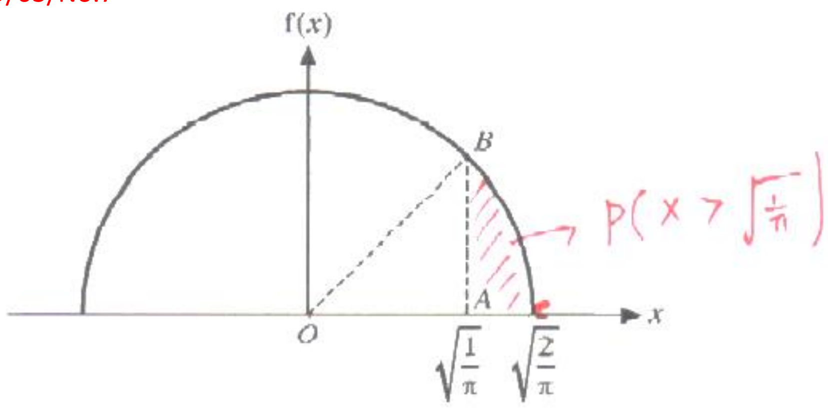
$$= \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{3}{2}\left[\left(\frac{1^2}{2} - \frac{1^4}{4}\right) - 0\right]$$

$$= \frac{3}{2}\left(\frac{1}{4}\right)$$

$$= \frac{3}{8}$$

3. June/2023/Paper_9709/63/No.7



A random variable X has probability density function f, where the graph of $y = \mathrm{f}(x)$ is a semicircle with centre (0, 0) and radius $\sqrt{\frac{2}{\pi}}$, entirely above the x-axis. Elsewhere $\mathrm{f}(x) = 0$ (see diagram).

(a) Verify that f can be a probability density function. [2]

If f is a Probability density function then the area under the graph must be equal to 1.

Area of the semicircle $= \frac{1}{2}\pi r^2$

$= \frac{1}{2} \times \pi \times \left(\sqrt{\frac{2}{\pi}}\right)^2$

$= \frac{1}{\cancel{2}} \times \cancel{\pi} \times \frac{\cancel{2}}{\cancel{\pi}}$

$= 1$

$f(x) \geq 0$ So f can be a probability density function

A and B are the points where the line $x = \sqrt{\frac{1}{\pi}}$ meets the x-axis and the semicircle respectively.

(b) Show that angle AOB is $\frac{1}{4}\pi$ radians and hence find $\mathrm{P}\left(X > \sqrt{\frac{1}{\pi}}\right)$. [6]

OB is the radius $= \sqrt{\frac{2}{\pi}}$

Using trigonometric ratios :

$\cos \angle AOB = \dfrac{\sqrt{\frac{1}{\pi}}}{\sqrt{\frac{2}{\pi}}}$

$\cos \angle AOB = \sqrt{\frac{1}{\pi}} \times \sqrt{\frac{\pi}{2}} = \sqrt{\frac{1}{\cancel{\pi}} \times \frac{\cancel{\pi}}{2}} = \frac{1}{\sqrt{2}}$

$\angle AOB = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$\therefore \angle AOB = \frac{1}{4}\pi$ radians.

$P\left(X > \sqrt{\frac{1}{\pi}}\right) =$ Area of the sector OBC. $-$ Area of the triangle OAB.

Area of the sector $= \frac{1}{2} r^2 \theta$

$= \frac{1}{2} \times \left(\sqrt{\frac{2}{\pi}}\right)^2 \times \frac{1}{4}\pi$

$= \frac{1}{2} \times \frac{2}{\cancel{\pi}} \times \frac{1}{4}\cancel{\pi} = \frac{1}{4}$

Area of the triangle $= \frac{1}{2} \times OA \times OB \times \sin \angle AOB$

$= \frac{1}{2} \times \sqrt{\frac{1}{\pi}} \times \sqrt{\frac{2}{\pi}} \times \sin\left(\frac{1}{4}\pi\right)$

$= \frac{1}{2} \times \sqrt{\frac{2}{\pi^2}} \times \frac{1}{\sqrt{2}}$

$= \frac{1}{2} \times \frac{\cancel{\sqrt{2}}}{\pi} \times \frac{1}{\cancel{\sqrt{2}}} = \frac{1}{2\pi}$

$\Rightarrow P\left(X > \sqrt{\frac{1}{\pi}}\right) = \frac{1}{4} - \frac{1}{2\pi} = 0.0908$ (3s.f).