Differentiation – 2023 June A2 Math 9709

1. June/2023/Paper_9709/21/No.2

A curve has equation $y = \dfrac{2 + 3\ln x}{1 + 2x}$.

Find the equation of the tangent to the curve at the point $(1, \frac{2}{3})$. Give your answer in the form $ax + by + c = 0$, where a, b and c are integers. [5]

Gradient of the tangent $= \dfrac{dy}{dx}$

Using quotient rule $\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Let $u = 2 + 3\ln x$

$\dfrac{du}{dx} = \dfrac{3}{x}$

Let $v = 1 + 2x$

$\dfrac{dv}{dx} = 2$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{(1+2x)\left(\frac{3}{x}\right) - (2 + 3\ln x)2}{(1+2x)^2}$

At $(1, \frac{2}{3})$, $\left.\dfrac{dy}{dx}\right|_{x=1} = \dfrac{(1+2)(3) - (2 + 3\ln 1)2}{(1+2)^2}$

$= \dfrac{9-4}{9} = \dfrac{5}{9}$

Equation of the tangent at $\left(1, \frac{2}{3}\right)$ is :

$$\left(y - \frac{2}{3} = \frac{5}{9}(x-1)\right) \times 9$$

$$9y - 6 = 5(x-1)$$

$$9y - 6 = 5x - 5$$

$$\Rightarrow 5x - 5 - 9y + 6 = 0$$

$$\therefore 5x - 9y + 1 = 0$$

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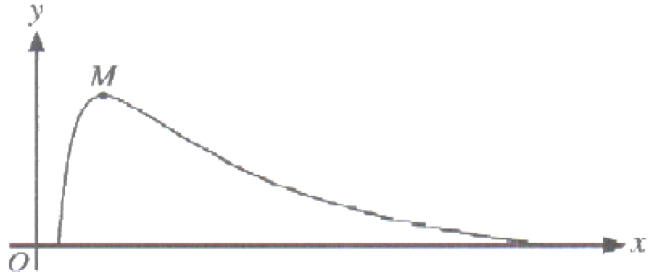
2. June/2023/Paper_9709/21/No.5



The diagram shows the curve with parametric equations

$$x = 4e^{2t}, \qquad y = 5e^{-t}\cos 2t,$$

for $-\frac{1}{4}\pi \leqslant t \leqslant \frac{1}{4}\pi$. The curve has a maximum point M.

(a) Find an expression for $\frac{dy}{dx}$ in terms of t. **[3]**

Using chain rule : $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ or $\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$x = 4e^{2t}$

$\frac{dx}{dt} = 4e^{2t} \cdot \frac{d}{dt}(2t) = 4e^{2t}(2) = 8e^{2t}$

$y = 5e^{-t}\cos 2t$ (using product rule)

$\frac{dy}{dt} = 5\left[\cos 2t \cdot \frac{d}{dt}(e^{-t}) + e^{-t} \cdot \frac{d}{dt}(\cos 2t)\right]$

$= 5\left[\cos 2t\,(e^{-t}) \cdot \frac{d}{dt}(-t) + e^{-t}(-\sin 2t) \cdot \frac{d}{dt}(2t)\right]$

$= 5\left[\cos 2t\,(e^{-t})(-1) + e^{-t}(-\sin 2t)(2)\right]$

$= 5\left[-e^{-t}\cos 2t - 2e^{-t}\sin 2t\right]$

$= -5e^{-t}\cos 2t - 10e^{-t}\sin 2t$

$\Rightarrow \frac{dy}{dx} = \frac{-5e^{-t}\cos 2t - 10e^{-t}\sin 2t}{8e^{2t}}$

(b) Find the coordinates of M, giving each coordinate correct to 3 significant figures. [5]

For maximum points, set $\frac{dy}{dx} = 0$

$\Rightarrow \frac{-5e^{-t}\cos 2t - 10e^{-t}\sin 2t}{8e^{2t}} = 0$

$\Rightarrow -5e^{-t}\cos 2t - 10e^{-t}\sin 2t = 0$

$5e^{-t}(-\cos 2t - 2\sin 2t) = 0$, but $e^{-t} > 0$ for all t.

$\Rightarrow -\cos 2t - 2\sin 2t = 0$

$-\cos 2t = 2\sin 2t$

$-\frac{1}{2} = \frac{\sin 2t}{\cos 2t} \Rightarrow \tan 2t = -\frac{1}{2}$

$2t = \tan^{-1}\left(-\frac{1}{2}\right)$

$t = \frac{1}{2}\tan^{-1}\left(-\frac{1}{2}\right) = -0.231$ (3s.f)

$x = 4e^{2t}$

$x = 4e^{2(-0.231)}$

$= 4e^{-0.462}$

$\therefore x = 2.52$ (3s.f)

$y = 5e^{-t}\cos 2t$

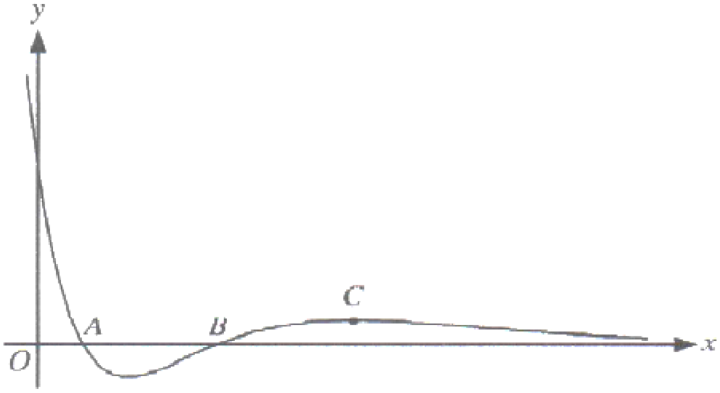
$= 5e^{-(-0.231)}\cos(2 \times -0.231)$

$= 5e^{0.231}\cos(-0.462)$

$= 5.64$ (3s.f)

$\therefore M(2.52, 5.64)$

3. June/2023/Paper_9709/22/No.5



The diagram shows the curve with equation $y = e^{-\frac{1}{2}x}(x^2 - 5x + 4)$. The curve crosses the x-axis at the points A and B, and has a maximum at the point C.

(a) Find the exact gradient of the curve at B. **[5]**

Gradient $= \dfrac{dy}{dx}$

Using product rule:

$$\frac{dy}{dx} = (x^2 - 5x + 4) \cdot \frac{d}{dx}\left(e^{-\frac{1}{2}x}\right) + e^{-\frac{1}{2}x} \cdot \frac{d}{dx}(x^2 - 5x + 4)$$

$$= (x^2 - 5x + 4)\left(e^{-\frac{1}{2}x} \cdot \frac{d}{dx}\left(-\frac{1}{2}x\right)\right) + e^{-\frac{1}{2}x}(2x - 5)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{2}e^{-\frac{1}{2}x}(x^2 - 5x + 4) + e^{-\frac{1}{2}x}(2x - 5)$$

When the curve crosses the x-axis, $y = 0$

$$\Rightarrow e^{-\frac{1}{2}x}(x^2 - 5x + 4) = 0$$

But $e^{-\frac{1}{2}x} > 0$ for all x

So $x^2 - 5x + 4 = 0$

$(x - 1)(x - 4) = 0$

$\Rightarrow x = 1, 4$

$\therefore$ x - Coordinate of $A = 1$

x - Coordinate of $B = 4$.

When $x = 4$

$$\frac{dy}{dx} = -\frac{1}{2}e^{-\frac{1}{2}(4)}\left(4^2 - 5(4) + 4\right) + e^{-\frac{1}{2}(4)}\left(2(4) - 5\right)$$

$$= -\frac{1}{2}e^{-2}(0) + e^{-2}(3)$$

$$\therefore \frac{dy}{dx} = 3e^{-2}$$

(b) Find the exact coordinates of C. [4]

C is the maximum point of the curve.

For maximum points, set $\frac{dy}{dx} = 0$

$\Rightarrow$ From part (a) $\frac{dy}{dx} = -\frac{1}{2} e^{-\frac{1}{2}x}(x^2 - 5x + 4) + e^{-\frac{1}{2}x}(2x - 5) = 0$

$\Rightarrow e^{-\frac{1}{2}x}\left(-\frac{1}{2}(x^2 - 5x + 4) + 2x - 5\right) = 0$

But $e^{-\frac{1}{2}x} > 0$ for all x, so $-\frac{1}{2}(x^2 - 5x + 4) + 2x - 5 = 0$

$2\left(-\frac{1}{2}(x^2 - 5x + 4) + 2x - 5 = 0\right)$

$-(x^2 - 5x + 4) + 4x - 10 = 0$

$-x^2 + 5x - 4 + 4x - 10 = 0$

$-x^2 + 9x - 14 = 0 \Rightarrow x^2 - 9x + 14 = 0$

$(x - 2)(x - 7) = 0$

So $x = 2, 7$

But x - coordinate of $C > 4$, so $x = 7$

$y = e^{-\frac{1}{2}x}(x^2 - 5x + 4)$

Substitute $x = 7$

$\Rightarrow y = e^{-\frac{1}{2}(7)}\left(7^2 - 5(7) + 4\right)$

$y = e^{-\frac{7}{2}}(18) = 18e^{-\frac{7}{2}}$

$\therefore C\left(7, 18e^{-\frac{7}{2}}\right)$