1. Nov/2023/Paper_9709/21/No.2

A curve has equation $y = 3\tan\frac{1}{2}x\cos 2x$.

Find the gradient of the curve at the point for which $x = \frac{1}{3}\pi$. [5]

Gradient of the Curve $= \left.\frac{dy}{dx}\right|_{x=\frac{1}{3}\pi}$

$\frac{dy}{dx} = \frac{d}{dx}\left(3\tan\frac{1}{2}x\cos 2x\right)$ (using product rule)

$= 3\left[\cos 2x \cdot \frac{d}{dx}\left(\tan\frac{1}{2}x\right) + \tan\frac{1}{2}x \cdot \frac{d}{dx}(\cos 2x)\right]$

$= 3\left[\cos 2x\left(\sec^2\frac{1}{2}x\right) \cdot \frac{d}{dx}\left(\frac{1}{2}x\right) + \tan\frac{1}{2}x \cdot -\sin 2x \cdot \frac{d}{dx}(2x)\right]$

$= 3\left[\cos 2x\left(\sec^2\frac{1}{2}x\right)\left(\frac{1}{2}\right) + \tan\frac{1}{2}x(-\sin 2x)(2)\right]$

$= \frac{3}{2}\cos 2x\sec^2\frac{1}{2}x - 6\tan\frac{1}{2}x\sin 2x$

But $\sec^2\frac{1}{2}x = \frac{1}{\cos^2\frac{1}{2}x}$

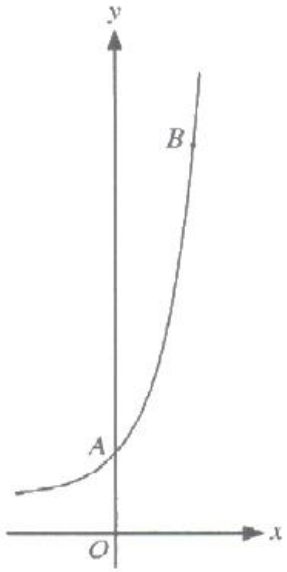
$= \frac{3\cos 2x}{2\cos^2\frac{1}{2}x} - 6\tan\frac{1}{2}x\sin 2x$

When $x = \frac{1}{3}\pi$, $\frac{dy}{dx} = \frac{3\cos\left(2 \times \frac{1}{3}\pi\right)}{2\left(\cos\frac{1}{2}\left(\frac{1}{3}\pi\right)\right)^2} - 6\tan\left(\frac{1}{2} \times \frac{1}{3}\pi\right)\sin\left(2 \times \frac{1}{3}\pi\right)$

$= \frac{3(-0.5)}{2\left(\frac{\sqrt{3}}{2}\right)^2} - 6\left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right)$

$= -1 - 3 = -4$

2. Nov/2023/Paper_9709/22/No.6



The diagram shows the curve with parametric equations

$$x = 3\ln(2t - 3), \qquad y = 4t\ln t.$$

The curve crosses the y-axis at the point A. At the point B, the gradient of the curve is 12.

(a) Find the exact gradient of the curve at A. **[5]**

When the curve crosses the y-axis, $x = 0$

$\Rightarrow \frac{3\ln(2t-3)}{3} = \frac{0}{3}$

$\ln(2t-3) = 0$

Introduce exponentials

$e^{\ln(2t-3)} = e^{0}$

$2t - 3 = 1$

$\frac{2t}{2} = \frac{4}{2} \Rightarrow t = 2$

Gradient $= \frac{dy}{dx}$

Using chain rule:

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

Using product rule:

$\frac{dy}{dt} = 4\left[\ln t \cdot \frac{d}{dt}(t) + t \cdot \frac{d}{dt}\ln t\right]$

$= 4\left[\ln t + t\left(\frac{1}{t}\right)\right]$

$= 4\ln t + 4$

$\frac{dx}{dt} = \frac{3}{2t-3} \cdot \frac{d}{dt}(2t-3)$

$= \frac{3}{2t-3}(2) = \frac{6}{2t-3}$

$\Rightarrow \frac{dy}{dx} = (4\ln t + 4) \times \left(\frac{1}{\frac{6}{2t-3}}\right)$

$= \frac{(4\ln t + 4)(2t-3)}{6}$

At $t = 2$, $\frac{dy}{dx} = \frac{(4\ln 2 + 4)(2(2)-3)}{6}$

$= \frac{4\ln 2 + 4}{6}$

$= \frac{2}{3}\ln 2 + \frac{2}{3}$

(b) Show that the value of the parameter t at B satisfies the equation

$$t = \frac{9}{1+\ln t} + \frac{3}{2}.$$ [2]

At B $\frac{dy}{dx} = 12$

From (a) $\frac{dy}{dx} = \frac{(4\ln t + 4)(2t-3)}{6} = 12$

$\Rightarrow (4\ln t + 4)(2t-3) = 6 \times 12$

$\frac{(4\ln t + 4)(2t-3)}{4\ln t + 4} = \frac{72}{4\ln t + 4}$

$2t - 3 = \frac{72}{4(\ln t + 1)} \Rightarrow 2t - 3 = \frac{18}{\ln t + 1}$

$2t = \frac{18}{\ln t + 1} + 3 \Rightarrow t = \frac{1}{2}\left(\frac{18}{\ln t + 1} + 3\right)$

$\therefore t = \frac{9}{1+\ln t} + \frac{3}{2}$ As required.

(c) Use an iterative formula, based on the equation in (b), to find the value of t at B, giving your answer correct to 3 significant figures. Use an initial value of 5 and give the result of each iteration to 5 significant figures. [3]

Iterative formular: $t_{n+1} = \frac{9}{1+\ln t_n} + \frac{3}{2}$, $t_0 = 5$

$t_1 = \frac{9}{1+\ln 5} + \frac{3}{2} = 4.9490$

$t_2 = \frac{9}{1+\ln 4.9490} + \frac{3}{2} = 4.9626$

$t_3 = \frac{9}{1+\ln 4.9626} + \frac{3}{2}$

$= 4.9590$

t converges at t_2 correct to (3s.f)

$\therefore t = 4.96$

3. Nov/2023/Paper_9709/31/No.1

Find the exact coordinates of the points on the curve $y = \frac{x^2}{1-3x}$ at which the gradient of the tangent is equal to 8. [5]

Gradient $= \frac{dy}{dx} = 8$

Using quotient rule:

$\frac{dy}{dx} = \frac{V\frac{du}{dx} - U\frac{dv}{dx}}{V^2}$

let $U = x^2$

$\frac{du}{dx} = 2x$

$8[1(1-3x)-3x(1-3x)] = 2x-3x^2$

$8[1-3x-3x+9x^2] = 2x-3x^2$

$8[1-6x+9x^2] = 2x-3x^2$

$8-48x+72x^2-2x+3x^2=0$

$75x^2-50x+8=0$

Solving for x using the Quadratic formular:

$x = \frac{-(-50) \pm \sqrt{(-50)^2-4(75)(8)}}{2\times 75}$

Let $V = 1 - 3x$

$\frac{dV}{dx} = -3$

$\frac{dy}{dx} = \frac{(1-3x)(2x) - x^2(-3)}{(1-3x)^2}$

$= \frac{2x - 6x^2 + 3x^2}{(1-3x)^2}$

$= \frac{2x - 3x^2}{(1-3x)^2}$

But $\frac{dy}{dx} = 8$

$\Rightarrow \frac{2x - 3x^2}{(1-3x)^2} = 8$

$\Rightarrow 8(1-3x)^2 = 2x - 3x^2$

$= \frac{50 \pm 10}{150}$

$x = \frac{50-10}{150} = \frac{4}{15}$, $x = \frac{50+10}{150} = \frac{2}{5}$

When $x = \frac{4}{15}$, $y = \frac{\left(\frac{4}{15}\right)^2}{1 - 3\left(\frac{4}{15}\right)}$

$= \frac{16}{45}$

When $x = \frac{2}{5}$, $y = \frac{\left(\frac{2}{5}\right)^2}{1 - 3\left(\frac{2}{5}\right)}$

$= -\frac{4}{5}$

$\therefore$ Coordinates are :

$\left(\frac{4}{15}, \frac{16}{45}\right)$ and $\left(\frac{2}{5}, -\frac{4}{5}\right)$

4. Nov/2023/Paper_9709/31/No.6

The parametric equations of a curve are

$$x = \sqrt{t} + 3, \qquad y = \ln t,$$

for $t > 0$.

(a) Obtain a simplified expression for $\frac{dy}{dx}$ in terms of t. [3]

Using chain rule : $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$x = t^{\frac{1}{2}} + 3$

$\frac{dx}{dt} = \frac{1}{2} t^{-\frac{1}{2}} = \frac{1}{2\sqrt{t}}$

$y = \ln t$

$\frac{dy}{dt} = \frac{1}{t}$

$\frac{dy}{dx} = \frac{1}{t} \times \frac{1}{\frac{1}{2\sqrt{t}}} = \frac{2\sqrt{t}}{t}$

$\frac{dy}{dx} = \frac{2\sqrt{t}}{t} \times \frac{\sqrt{t}}{\sqrt{t}} = \frac{2\cancel{t}}{\cancel{t}\sqrt{t}} = \frac{2}{\sqrt{t}}$

(b) Hence find the exact coordinates of the point on the curve at which the gradient of the normal is -2. [3]

Gradient of the tangent $= \frac{dy}{dx}$

$m_T = \frac{dy}{dx} = \frac{2}{\sqrt{t}}$

$m_N = \frac{-1}{m_T} = \frac{-1}{\frac{2}{\sqrt{t}}} = \frac{+\sqrt{t}}{2} = -2$

$\Rightarrow \sqrt{t} = 2 \times 2 = 4$

$t = 4^2 = 16$

$\therefore x = 4 + 3 = 7$

$y = \ln 16$

$\therefore$ The coordinates are:

$(7, \ln 16)$

5. Nov/2023/Paper_9709/32/No.2

The parametric equations of a curve are

$$x = (\ln t)^2, \qquad y = e^{2-t^2},$$

for $t > 0$.

Find the gradient of the curve at the point where $t = e$, simplifying your answer. [4]

Gradient $= \frac{dy}{dx}$

Using chain rule : $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ or $\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$y = e^{2-t^2}$ (using product rule)

$\frac{dy}{dt} = e^{2-t^2} \cdot \frac{d}{dt}(2-t^2)$

$= e^{2-t^2}(-2t) = -2t\,e^{2-t^2}$

$x = (\ln t)^2$

$\frac{dx}{dt} = 2(\ln t)^{(2-1)} \cdot \frac{d}{dt}(\ln t)$

$= 2\ln t \cdot \frac{1}{t} = \frac{2}{t}\ln t$

$\Rightarrow \frac{dy}{dx} = -2t\,e^{2-t^2} \times \frac{1}{\frac{2}{t}\ln t} = -\cancel{2}t\,e^{2-t^2} \times \frac{t}{\cancel{2}\ln t}$

$= \frac{-t^2 e^{2-t^2}}{\ln t}$

When $t = e$, $\frac{dy}{dx} = \frac{-e^2\left(e^{2-e^2}\right)}{\ln e}$, but $\ln e = 1$

$= -e^2\left(e^{2-e^2}\right)$ (using laws of indices)

$= -\left(e^{2+2-e^2}\right)$

$= -e^{4-e^2}$

6. Nov/2023/Paper_9709/33/No.5

Find the exact coordinates of the stationary points of the curve $y = \dfrac{e^{3x^2-1}}{1-x^2}$. [6]

For Stationary points, set $\frac{dy}{dx} = 0$

Using quotient rule:

$$\frac{dy}{dx} = \frac{V\frac{du}{dx} - u\frac{dv}{dx}}{V^2}$$

Let $u = e^{3x^2-1}$

$$\frac{du}{dx} = e^{3x^2-1} \cdot \frac{d}{dx}(3x^2-1)$$

$$= e^{3x^2-1} \cdot (6x)$$

$$= 6x\, e^{3x^2-1}$$

Let $V = 1 - x^2$

$$\frac{dv}{dx} = -2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1-x^2)(6x e^{3x^2-1}) - (e^{3x^2-1})(-2x)}{(1-x^2)^2}$$

$$= \frac{e^{3x^2-1}\left(6x(1-x^2) - (-2x)\right)}{(1-x^2)^2} = 0$$

$$\Rightarrow e^{3x^2-1}(6x - 6x^3 + 2x) = 0$$

$$e^{3x^2-1}(8x - 6x^3) = 0$$

But $e^{3x^2-1} > 0$ for all x

$$\Rightarrow 8x - 6x^3 = 0$$

$$2x(4 - 3x^2) = 0$$

$x = 0$, $4 - 3x^2 = 0$

$x = 0$, $x^2 = \frac{4}{3} \Rightarrow x = \pm\sqrt{\frac{4}{3}}$

$$x = \pm\frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \pm\frac{2\sqrt{3}}{3}$$

When $x = 0$, $y = \dfrac{e^{0-1}}{1-0} = e^{-1}$

When $x = \pm\dfrac{2\sqrt{3}}{3}$,

$$y = \frac{e^{3\left(\pm\frac{2\sqrt{3}}{3}\right)^2 - 1}}{1 - \left(\pm\frac{2\sqrt{3}}{3}\right)^2}$$

$$= \frac{e^{3\left(\frac{4}{3}\right)-1}}{1 - \frac{4}{3}} = \frac{e^3}{-\frac{1}{3}}$$

$$= -3e^3$$

$\therefore$ The Coordinates are: $(0, e^{-1})$, $\left(-\frac{2\sqrt{3}}{3}, -3e^3\right)$ and $\left(\frac{2\sqrt{3}}{3}, -3e^3\right)$

7. Nov/2023/Paper_9709/33/No.7

The equation of a curve is $x^3 + y^2 + 3x^2 + 3y = 4$.

(a) Show that $\dfrac{dy}{dx} = -\dfrac{3x^2 + 6x}{2y + 3}$. [3]

Using implicit differentiation

$$\Rightarrow 3x^2 + 2y\frac{dy}{dx} + 6x + 3\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} + 3\frac{dy}{dx} = -3x^2 - 6x$$

$$\frac{dy}{dx}\frac{\cancel{(2y+3)}}{\cancel{2y+3}} = \frac{-(3x^2+6x)}{2y+3}$$

$$\therefore \frac{dy}{dx} = -\frac{3x^2+6x}{2y+3}$$

(b) Hence find the coordinates of the points on the curve at which the tangent is parallel to the x-axis. [5]

The gradient of the tangent $= \frac{dy}{dx} = 0$

From (a) $\frac{dy}{dx} = -\frac{3x^2+6x}{2y+3} = 0$

$\Rightarrow -(3x^2+6x) = 0$

$-3x(x+2) = 0$

$-3x = 0, \quad x+2 = 0$

$\therefore x = 0, \quad x = -2$

Substitute x into the equation of the curve.

When $x=0$, $0 + y^2 + 0 + 3y = 4$

$\Rightarrow y^2 + 3y - 4 = 0$

$(y-1)(y+4) = 0$

$y-1 = 0, \quad y+4 = 0$

$\therefore y = 1, \quad y = -4$

When $x = -2$, $(-2)^3 + y^2 + 3(-2)^2 + 3y = 4$

$\Rightarrow -8 + y^2 + 12 + 3y - 4 = 0$

$y^2 + 3y = 0$

$y(y+3) = 0$

$y = 0, \quad y+3 = 0$

$\therefore y = 0, \quad y = -3$

$\therefore$ The Coordinates are: $(0, 1)$, $(0, -4)$, $(-2, 0)$ and $(-2, -3)$.