Trigonometry – 2023 June AS Math 9709

1. June/2023/Paper_9709/11/No.1

Solve the equation $4\sin\theta + \tan\theta = 0$ for $0° < \theta < 180°$. [3]

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$\Rightarrow \left(4\sin\theta + \dfrac{\sin\theta}{\cos\theta} = 0\right) \times \cos\theta$

$\Rightarrow 4\sin\theta\cos\theta + \sin\theta = 0$

$\sin\theta(4\cos\theta + 1) = 0$

$\Rightarrow \sin\theta = 0, \quad 4\cos\theta + 1 = 0$

$\sin\theta = 0, \quad \cos\theta = -\frac{1}{4}$

When $\sin\theta = 0$, $\theta = \sin^{-1}(0)$

$\theta = 0, \pi$

When $\cos\theta = -\frac{1}{4}$, $\theta = \cos^{-1}\left(-\frac{1}{4}\right)$

$\theta = 104.5°, (360° - 104.5°)$

$= 104.5°, 255.5°$

For $0 < \theta < 180°$

$\therefore \theta = 104.5°$

2. June/2023/Paper_9709/11/No.7

A curve has equation $y = 2 + 3\sin\frac{1}{2}x$ for $0 \leqslant x \leqslant 4\pi$.

(a) State greatest and least values of y. [2]

Range of the sine function is $[-1, 1]$

For Greatest value, $\sin\frac{1}{2}x = 1 \Rightarrow y = 2 + 3(1) = 5$

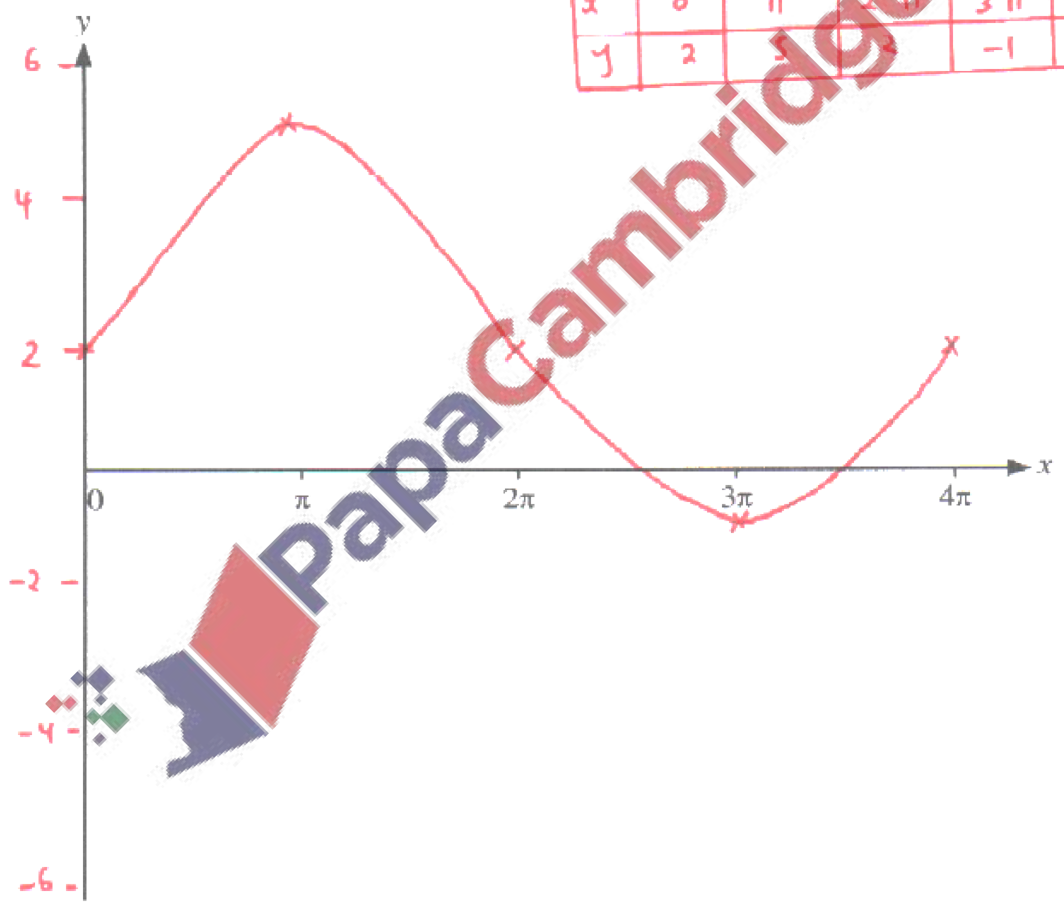
For least value, $\sin\frac{1}{2}x = -1 \Rightarrow y = 2 + 3(-1)$

$= -1$

(b) Sketch the curve. [2]

Table of Values

x	0	π	2π	3π	4π
y	2	5	2	-1	2



(c) State the number of solutions of the equation

$$2 + 3\sin\tfrac{1}{2}x = 5 - 2x$$

for $0 \leqslant x \leqslant 4\pi$. [1]

The point of intersection of $y = 2 + 3\sin\frac{1}{2}x$

and $y = 5 - 2x$.

$\therefore$ Number of solutions $= 1$

3. June/2023/Paper_9709/12/No.7

(a) (i) By first expanding $(\cos\theta + \sin\theta)^2$, find the three solutions of the equation

$$(\cos\theta + \sin\theta)^2 = 1$$

for $0 \leqslant \theta \leqslant \pi$. [3]

$(\cos\theta + \sin\theta)^2 = \cos\theta(\cos\theta + \sin\theta) + \sin\theta(\cos\theta + \sin\theta)$

$= \cos^2\theta + \sin\theta\cos\theta + \sin\theta\cos\theta + \sin^2\theta$

$= (\cos^2\theta + \sin^2\theta) + 2\sin\theta\cos\theta$

But $\cos^2\theta + \sin^2\theta = 1$, $\sin 2\theta = 2\sin\theta\cos\theta$

$\Rightarrow (\cos\theta + \sin\theta)^2 = 1 + \sin 2\theta$

$(\cos\theta + \sin\theta)^2 = 1$

$\Rightarrow 1 + \sin 2\theta = 1 \Rightarrow \sin 2\theta = 0$

$2\theta = \sin^{-1}(0)$

$2\theta = 0, \pi, 2\pi$

$\theta = \frac{1}{2}(0, \pi, 2\pi)$

$\therefore \theta = 0, \frac{\pi}{2}, \pi$

(ii) Hence verify that the only solutions of the equation $\cos\theta + \sin\theta = 1$ for $0 \leqslant \theta \leqslant \pi$ are 0 and $\frac{1}{2}\pi$. [2]

When $\theta = 0$, $\cos 0 + \sin 0 = 1 + 0 = 1$

$\theta = \frac{\pi}{2}$, $\cos\frac{\pi}{2} + \sin\frac{\pi}{2} = 0 + 1 = 1$

When $\theta = \pi$, $\cos\pi + \sin\pi = -1 + 0$

$= -1 \neq 1$

So the only solutions are 0 and $\frac{\pi}{2}$.

(b) Prove the identity $\dfrac{\sin\theta}{\cos\theta+\sin\theta}+\dfrac{1-\cos\theta}{\cos\theta-\sin\theta}=\dfrac{\cos\theta+\sin\theta-1}{1-2\sin^2\theta}$. [3]

Consider the LHS:

$$\frac{\sin\theta}{\cos\theta+\sin\theta}+\frac{1-\cos\theta}{\cos\theta-\sin\theta}=\frac{\sin\theta(\cos\theta-\sin\theta)+(\cos\theta+\sin\theta)(1-\cos\theta)}{(\cos\theta+\sin\theta)(\cos\theta-\sin\theta)}$$

$$=\frac{\sin\theta\cos\theta-\sin^2\theta+1(\cos\theta+\sin\theta)-\cos\theta(\cos\theta+\sin\theta)}{\cos\theta(\cos\theta-\sin\theta)+\sin\theta(\cos\theta-\sin\theta)}$$

$$=\frac{\cancel{\sin\theta\cos\theta}-\sin^2\theta+\cos\theta+\sin\theta-\cos^2\theta-\cancel{\sin\theta\cos\theta}}{\cos^2\theta-\cancel{\sin\theta\cos\theta}+\cancel{\sin\theta\cos\theta}-\sin^2\theta}$$

$$=\frac{\cos\theta+\sin\theta-(\sin^2\theta+\cos^2\theta)}{\cos^2\theta-\sin^2\theta}\quad \sin^2\theta+\cos^2\theta=1$$

$$=\frac{\cos\theta+\sin\theta-1}{1-\sin^2\theta-\sin^2\theta}=\frac{\cos\theta+\sin\theta-1}{1-2\sin^2\theta}$$

(c) Using the results of **(a)(ii)** and **(b)**, solve the equation

$$\frac{\sin\theta}{\cos\theta+\sin\theta}+\frac{1-\cos\theta}{\cos\theta-\sin\theta}=2(\cos\theta+\sin\theta-1)$$

for $0\leqslant\theta\leqslant\pi$. [3]

From part (b) $\dfrac{\sin\theta}{\cos\theta+\sin\theta}+\dfrac{1-\cos\theta}{\cos\theta-\sin\theta}=\dfrac{\cos\theta+\sin\theta-1}{1-2\sin^2\theta}$

$\Rightarrow \dfrac{\cos\theta+\cancel{\sin\theta-1}}{1-2\sin^2\theta}=2(\cos\theta+\cancel{\sin\theta-1})$

$\Rightarrow \cos\theta+\sin\theta-1=0 \quad ....(i)$

$\dfrac{1}{1-2\sin^2\theta}=2$

$1-2\sin^2\theta=0.5$

$2\sin^2\theta=1-0.5=0.5$

$\Rightarrow \sin^2\theta=\dfrac{0.5}{2}=0.25$

$\sin\theta=\pm\sqrt{0.25}=\pm0.5$

$\theta=\sin^{-1}(\pm0.5)=\dfrac{\pi}{6},\ \dfrac{5\pi}{6}$

Solving (i)

$\Rightarrow \cos\theta+\sin\theta=1$

$(\cos\theta+\sin\theta)^2=1^2$

$\underbrace{\cos^2\theta+\sin^2\theta}_{=1}+2\sin\theta\cos\theta=1$

$2\sin\theta\cos\theta=1-1=0$

$\sin2\theta=0$

$\theta=\dfrac{1}{2}\sin^{-1}(0)=\dfrac{1}{2}(0,\pi)$

$=0,\ \dfrac{1}{2}\pi$

$\therefore \theta=0,\ \dfrac{1}{6}\pi,\ \dfrac{1}{2}\pi,\ \dfrac{5}{6}\pi$

4. June/2023/Paper_9709/13/No.4

(a) Show that the equation

$$3\tan^2 x - 3\sin^2 x - 4 = 0$$

may be expressed in the form $a\cos^4 x + b\cos^2 x + c = 0$, where a, b and c are constants to be found. [3]

Recall $\tan x = \dfrac{\sin x}{\cos x} \Rightarrow \tan^2 x = \dfrac{\sin^2 x}{\cos^2 x}$

$\left(3\dfrac{\sin^2 x}{\cos^2 x} - 3\sin^2 x - 4 = 0\right) \times \cos^2 x$

$\Rightarrow 3\sin^2 x - 3\sin^2 x\cos^2 x - 4\cos^2 x = 0$

But $\sin^2 x = 1 - \cos^2 x$

$\Rightarrow 3(1-\cos^2 x) - 3\cos^2 x(1-\cos^2 x) - 4\cos^2 x = 0$

$3 - 3\cos^2 x - 3\cos^2 x + 3\cos^4 x - 4\cos^2 x = 0$

$\therefore 3\cos^4 x - 10\cos^2 x + 3 = 0$.

$\Rightarrow a = 3,\ b = -10$ and $c = 3$

(b) Hence solve the equation $3\tan^2 x - 3\sin^2 x - 4 = 0$ for $0° \leqslant x \leqslant 180°$. [4]

From part (a)

$3\cos^4 x - 10\cos^2 x + 3 = 0$

Let $\cos^2 x = y$

$\Rightarrow 3y^2 - 10y + 3 = 0$

$(3y-1)(y-3) = 0$

$y = \dfrac{1}{3},\ 3$

$\Rightarrow \cos^2 x = \dfrac{1}{3},\ 3$

$\cos x = \pm\sqrt{\dfrac{1}{3}},\ \pm\sqrt{3}$

But the range of the cosine function is $[-1, 1]$ so

$\cos x = \pm\sqrt{3}$ is not defined.

$\therefore \cos x = \pm\dfrac{1}{\sqrt{3}}$

$x = \cos^{-1}\left(\pm\dfrac{1}{\sqrt{3}}\right)$

$\therefore x = 54.7°,\ 125.3°$