CAMBRIDGE INTERNATIONAL EXAMINATIONS
General Certificate of Education
Advanced Level

FURTHER MATHEMATICS **9231/01**

Paper 1

October/November 2003

3 hours

Additional materials: Answer Booklet/Paper
Graph paper
List of Formulae (MF10)

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

UNIVERSITY *of* CAMBRIDGE
Local Examinations Syndicate

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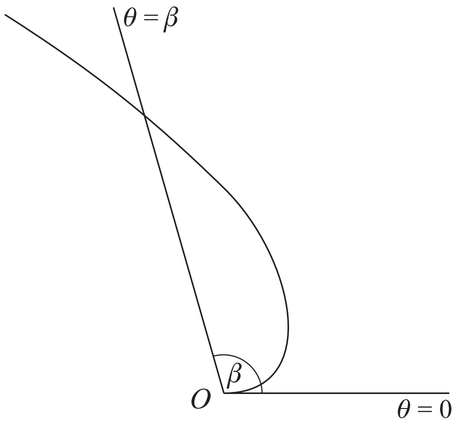

The curve C has polar equation

$$r = \theta^{\frac{1}{2}} e^{\theta^2/\pi},$$

where $0 \leqslant \theta \leqslant \pi$. The area of the finite region bounded by C and the line $\theta = \beta$ is π (see diagram). Show that

$$\beta = (\pi \ln 3)^{\frac{1}{2}}.$$ [6]

2 Given that

$$u_n = \frac{1}{n^2 - n + 1} - \frac{1}{n^2 + n + 1},$$

find $S_N = \sum_{n=N+1}^{2N} u_n$ in terms of N. [3]

Find a number M such that $S_N < 10^{-20}$ for all $N > M$. [3]

3 Three $n \times 1$ column vectors are denoted by $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{M}$ is an $n \times n$ matrix. Show that if $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ are linearly dependent then the vectors $\mathbf{Mx}_1$, $\mathbf{Mx}_2$, $\mathbf{Mx}_3$ are also linearly dependent. [2]

The vectors $\mathbf{y}_1$, $\mathbf{y}_2$, $\mathbf{y}_3$ and the matrix $\mathbf{P}$ are defined as follows:

$$\mathbf{y}_1 = \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}, \quad \mathbf{y}_2 = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}, \quad \mathbf{y}_3 = \begin{pmatrix} 5 \\ 51 \\ 55 \end{pmatrix},$$

$$\mathbf{P} = \begin{pmatrix} 1 & -4 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & -7 \end{pmatrix}.$$

(i) Show that $\mathbf{y}_1$, $\mathbf{y}_2$, $\mathbf{y}_3$ are linearly dependent. [2]

(ii) Find a basis for the linear space spanned by the vectors $\mathbf{Py}_1$, $\mathbf{Py}_2$, $\mathbf{Py}_3$. [2]

4 Given that $y = x \sin x$, find $\frac{d^2y}{dx^2}$ and $\frac{d^4y}{dx^4}$, simplifying your results as far as possible, and show that

$$\frac{d^6y}{dx^6} = -x \sin x + 6 \cos x.$$ [3]

Use induction to establish an expression for $\frac{d^{2n}y}{dx^{2n}}$, where n is a positive integer. [5]

5 The integral I_n is defined by

$$I_n = \int_0^{\frac{1}{4}\pi} \sec^n x \, dx.$$

By considering $\frac{d}{dx}(\tan x \sec^n x)$, or otherwise, show that

$$(n+1)I_{n+2} = 2^{\frac{1}{2}n} + nI_n.$$ [4]

Find the value of I_6. [4]

6 Find the sum of the squares of the roots of the equation

$$x^3 + x + 12 = 0,$$

and deduce that only one of the roots is real. [4]

The real root of the equation is denoted by α. Prove that $-3 < \alpha < -2$, and hence prove that the modulus of each of the other roots lies between 2 and $\sqrt{6}$. [5]

7 Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-\alpha t},$$

where α is a constant and $\alpha \neq 2$. [7]

Show that if $\alpha < 2$ then, whatever the initial conditions, $ye^{\alpha t} \to \frac{1}{(2-\alpha)^2}$ as $t \to \infty$. [2]

8 Given that $z = e^{i\theta}$ and n is a positive integer, show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta \qquad \text{and} \qquad z^n - \frac{1}{z^n} = 2i \sin n\theta.$$ [2]

Hence express $\sin^6 \theta$ in the form

$$p \cos 6\theta + q \cos 4\theta + r \cos 2\theta + s,$$

where the constants p, q, r, s are to be determined. [4]

Hence find the mean value of $\sin^6 \theta$ with respect to θ over the interval $0 \leqslant \theta \leqslant \frac{1}{4}\pi$. [5]

9 The line l_1 passes through the point A with position vector $\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and is parallel to the vector $3\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$. The variable line l_2 passes through the point $(1 + 5\cos t)\mathbf{i} - (1 + 5\sin t)\mathbf{j} - 14\mathbf{k}$, where $0 \leqslant t < 2\pi$, and is parallel to the vector $15\mathbf{i} + 8\mathbf{j} - 3\mathbf{k}$. The points P and Q are on l_1 and l_2 respectively, and PQ is perpendicular to both l_1 and l_2.

(i) Find the length of PQ in terms of t. [4]

(ii) Hence show that the lines l_1 and l_2 do not intersect, and find the maximum length of PQ as t varies. [3]

(iii) The plane Π_1 contains l_1 and PQ; the plane Π_2 contains l_2 and PQ. Find the angle between the planes Π_1 and Π_2, correct to the nearest tenth of a degree. [4]

10 Find the eigenvalues and corresponding eigenvectors of the matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 6 & 4 & 1 \\ -6 & -1 & 3 \\ 8 & 8 & 4 \end{pmatrix}. \qquad [8]$$

Hence find a non-singular matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 = \mathbf{PDP}^{-1}$. [4]

11 Answer only **one** of the following two alternatives.

EITHER

The curve C has equation $y = \dfrac{5(x-1)(x+2)}{(x-2)(x+3)}$.

(i) Express y in the form $P + \dfrac{Q}{x-2} + \dfrac{R}{x+3}$. [3]

(ii) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ for exactly one value of x and find the corresponding value of y. [4]

(iii) Write down the equations of all the asymptotes of C. [3]

(iv) Find the set of values of k for which the line $y = k$ does not intersect C. [4]

OR

A curve has equation $y = \frac{2}{3}x^{\frac{3}{2}}$, for $x \geqslant 0$. The arc of the curve joining the origin to the point where $x = 3$ is denoted by R.

(i) Find the length of R. [4]

(ii) Find the y-coordinate of the centroid of the region bounded by the x-axis, the line $x = 3$ and R. [5]

(iii) Show that the area of the surface generated when R is rotated through one revolution about the y-axis is $\frac{232}{15}\pi$. [5]