UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS
General Certificate of Education
Advanced Level

FURTHER MATHEMATICS **9231/01**

Paper 1

May/June 2004

3 hours

Additional materials: Answer Booklet/Paper
Graph paper
List of Formulae (MF10)

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

This document consists of **5** printed pages and **3** blank pages.

UNIVERSITY *of* CAMBRIDGE
International Examinations

1 Use the relevant standard results in the List of Formulae to prove that

$$S_N = \sum_{n=1}^{N}(8n^3 - 6n^2) = N(N+1)(2N^2 - 1).$$

Hence show that

$$\sum_{n=N+1}^{2N}(8n^3 - 6n^2)$$

can be expressed in the form

$$N(aN^3 + bN^2 + cN + d),$$

where the constants a, b, c, d are to be determined. [2]

2 The curve C has equation

$$y = \frac{x - ax^2}{x - 1},$$

where a is a constant and $a > 1$.

(i) Find the equations of the asymptotes of C. [3]

(ii) Show that the x-coordinates of both the turning points of C are positive. [4]

3 The curve C has equation

$$\left(x^2 + y^2\right)^2 = 4xy.$$

(i) Show that the polar equation of C is $r^2 = 2\sin 2\theta$. [2]

(ii) Draw a sketch of C, indicating any lines of symmetry as well as the form of C at the pole. [5]

(iii) Write down the maximum possible distance of a point of C from the pole. [1]

4 It is given that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{\ln x}{x}\right) = \frac{a_n \ln x + b_n}{x^{n+1}},$$

where a_n and b_n depend only on n.

(i) Find a_1, a_2 and a_3. [3]

(ii) Use mathematical induction to establish a formula for a_n. [5]

5 Write down the eigenvalues of the matrix **A**, given by

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 3 & -1 \\ 0 & 0 & 4 \end{pmatrix},$$

and obtain a set of corresponding eigenvectors. [6]

Find a non-singular matrix **P** and a diagonal matrix **D** such that $\mathbf{A}^5 = \mathbf{PDP}^{-1}$. [2]

6 Let

$$I_n = \int_{\mathrm{e}}^{\mathrm{e}^2} (\ln x)^n \, \mathrm{d}x,$$

where $n \geqslant 0$. By considering $\frac{\mathrm{d}}{\mathrm{d}x}\left[x(\ln x)^{n+1}\right]$, or otherwise, show that

$$I_{n+1} = 2^{n+1}\mathrm{e}^2 - \mathrm{e} - (n+1)I_n.$$ [4]

Find I_3 and deduce that the mean value of $(\ln x)^3$ over the interval $\mathrm{e} \leqslant x \leqslant \mathrm{e}^2$ is

$$2\left(\frac{\mathrm{e}+1}{\mathrm{e}-1}\right).$$ [5]

7 Find the roots of the equation

$$z^3 = -(4\sqrt{3}) + 4\mathrm{i},$$

giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 \leqslant \theta < 2\pi$. [5]

Denoting these roots by z_1, z_2, z_3, show that, for every positive integer k,

$$z_1^{3k} + z_2^{3k} + z_3^{3k} = 3\left(2^{3k}\mathrm{e}^{\frac{5}{6}k\pi\mathrm{i}}\right).$$ [4]

8 The curve C is defined parametrically by

$$x = t^3 - 3t, \quad y = 3t^2 + 1,$$

where $t > 1$.

(i) Show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is negative at every point of C. [5]

(ii) The arc of C joining the point where $t = 2$ to the point where $t = 3$ is rotated through one complete revolution about the x-axis. Find the area of the surface generated. [5]

9 The variable y depends on x and the variables x and t are related by $x = \frac{1}{t}$. Show that

$$\frac{dy}{dx} = -t^2\frac{dy}{dt} \quad \text{and} \quad \frac{d^2y}{dx^2} = t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt}.$$

The variables x and y are related by the differential equation

$$x^5\frac{d^2y}{dx^2} + (2x^4 - 5x^3)\frac{dy}{dx} + 4xy = 14x + 8.$$

Show that

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y = 8t + 14.$$ [2]

Hence find the general solution for y in terms of x. [5]

10 The linear transformation $\mathrm{T} : \mathbb{R}^4 \to \mathbb{R}^4$ is defined by

$$\mathrm{T} : \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \mapsto \mathbf{A}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix},$$

where

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 3 & -2 \\ 5 & 0 & 7 & -7 \\ 6 & 2 & 6 & \theta + 2 \\ 9 & 3 & 9 & \theta \end{pmatrix}.$$

(i) Show that when $\theta \neq -6$, the dimension of the null space K of T is 1, and that when $\theta = -6$, the dimension of K is 2. [4]

(ii) For the case $\theta \neq -6$, determine a basis vector $\mathbf{e}_1$ for K of the form $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ 0 \end{pmatrix}$, where x_1, y_1, z_1 are integers. [2]

(iii) For the case $\theta = -6$, determine a vector $\mathbf{e}_2$ of the form $\begin{pmatrix} x_2 \\ y_2 \\ 0 \\ t_2 \end{pmatrix}$, where x_2, y_2, t_2 are integers, such that $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis of K. [3]

(iv) Given that $\theta = -6$, $\mathbf{b} = \begin{pmatrix} 5 \\ 5 \\ 10 \\ 15 \end{pmatrix}$, $\mathbf{e}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, show that $\mathbf{x} = \mathbf{e}_0 + k_1\mathbf{e}_1 + k_2\mathbf{e}_2$ is a solution of the equation $\mathbf{Ax} = \mathbf{b}$ for all real values of k_1 and k_2. [3]

11 Answer only **one** of the following two alternatives.

EITHER

(i) Find the acute angle beween the line l whose equation is

$$\mathbf{r} = s(2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

and the plane Π_1 whose equation is

$$x - z = 0.$$ [3]

(ii) Find, in the form $ax + by + cz = 0$, the equation of the plane Π_2 which contains l and is perpendicular to Π_1. [3]

(iii) Find a vector equation of the line of intersection of the planes Π_1 and Π_2 and hence, or otherwise, show that the vectors $\mathbf{i} - \mathbf{k}$, $2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $3\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ are linearly dependent. [3]

(iv) The variable line m passes through the point with position vector $4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$ and is perpendicular to l. The line m meets Π_1 at Q. Find the minimum distance of Q from the origin, as m varies, giving your answer correct to 3 significant figures. [5]

OR

The roots of the equation

$$x^3 - x - 1 = 0$$

are α, β, γ, and

$$S_n = \alpha^n + \beta^n + \gamma^n.$$

(i) Use the relation $y = x^2$ to show that α^2, β^2, γ^2 are the roots of the equation

$$y^3 - 2y^2 + y - 1 = 0.$$ [3]

(ii) Hence, or otherwise, find the value of S_4. [2]

(iii) Find the values of S_8, S_{12} and S_{16}. [9]

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