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FOREWORD

This booklet contains reports written by Examiners on the work of candidates in certain papers. **Its contents are primarily for the information of the subject teachers concerned**.

FURTHER MATHEMATICS

GCE Advanced Level

Paper 9231/01
Paper 1

General comments

The overall quality of most of the work submitted in response to this paper was of a high standard and provided clear evidence of the majority of candidates being well prepared. Most candidates handed in some serious work in response to twelve questions. There were few very poor scripts and at the other extreme, those scripts whose total mark was above the upper quartile must be described as outstanding.

There was very little evidence of candidates running out of time. There were few misreads and almost no rubric infringement. On the negative side, one must remark that the very high levels of working accuracy that were much in evidence some years ago, and previously, were not maintained in this examination. The need for checking each stage of an evolving question response has been stressed repeatedly in previous A Level Further Mathematics reports. The consequences of not heeding this advice is clearly evident in scripts where the implementation of an effective strategy is negated by an elementary error early on in the response. The solution of a mathematical problem is a sequential process which once derailed has no prospect of reaching a successful conclusion.

Particular topics which were found to be difficult in the context of this paper were inequalities, induction, use of standard results to sum unfamiliar series, complex numbers, polar coordinates, and the obtaining of the general solution of a system of linear equations, though again, it must be said that more often than not, the main cause of failure was lack of a reliable technique with basic algebra. On the other hand, a lot of the work for the determination of asymptotes, curve sketching, the calculus topics generally and the eigenvalue problem was impressive.

Again the standard of presentation of work from some Centres was very satisfactory. From the rest, there were, in contrast, scripts which were very difficult to read and in which responses were fragmented and out of order.

Comments on specific questions

Question 1

There were many elementary working errors to be seen and a minority of candidates produced a completely correct response.

It was common for the first part of this question to be ignored. For the rest, many failed to obtain the correct argument of z^5 and thought the modulus of z^5 to be 16, or even –16. Many did not add multiples of 2π before dividing by 5. Very few used the first result, where this had been obtained, to find the roots. This is the most direct and least error prone strategy.

Answers: $e^{2\pi k i/5}$ where $k = 0, \ldots 4$; $2e^{\pi(6k+2)i/15}$ where $k = 0, \ldots, 4$.

Question 2

(i) The majority of responses began with something like 'Let H_k be the inductive hypothesis $u_k < 2$ for all positive integers k'. Such a statement is clearly meaningless and shows a fundamental lack of understanding of mathematical induction. Instead, the argument should begin with the equivalent of 'Let H_k be the inductive hypothesis $u_k < 2$ for some positive integer k'.

The proof that $u_k < 2 \Rightarrow u_{k+1} < 2$ appeared in most responses, but the conclusion of the inductive argument was frequently hazy. Thus, for example, many responses showed a validation of H_2, but then went on to claim H_n to be true for all $n \geqslant 1$.

(ii) A significant minority of candidates were unable to obtain a valid expansion of $\sqrt{9-\varepsilon}$ and so could make no progress. Actually a few used $9-\varepsilon = \left(3-\dfrac{\varepsilon}{6}\right)^2 - \dfrac{\varepsilon^2}{36} \approx \left(3-\dfrac{\varepsilon}{6}\right)^2$, or equivalent, in an intelligent way and so obtained the required result immediately. Generally, however, the average standard of responses to this part of the question was well below what had been expected.

Question 3

Most candidates obtained correct equations for both asymptotes. A common, erroneous result for the oblique asymptote was $y = x$ which undermined both sketch graphs. Nevertheless, the majority of graphs were correct and well drawn. In a small minority of cases, the curve drawn did not have a turning point at the origin or did not even pass through the origin.

Answers: $x = -\lambda$ and $y = x - \lambda$.

Question 4

This question was accurately answered by most candidates. Many errors occurred in the determination of the particular integral and/or the application of the given initial conditions.

Answer: $y = 3e^{-x} - 4e^{-2x} + 2e^{2x}$.

Question 5

Although the basic ideas involved were understood by a majority of candidates, only a small minority of responses were completely correct. Most working showed some lack of understanding of inequality arguments.

(i) Few candidates failed here.

(ii) Most responses started off with $\sum \alpha^2 = a^2 - 2b$. Beyond that it was not always made clear that all roots $> 1 \Rightarrow \sum \alpha^2 > 3$ which is an essential stage in establishing the required result.

(iii) Most responses got as far as showing that $S_3 = 3ab - a^3 - 3c$ and subsequently argued correctly to prove that $a^3 < 3ab - 3c - 3$ (*). For the rest, it is necessary to explain why $b > 0$ and not merely to assume that it must be the case. This apparently inconsequential inequality, which was alluded to by a small minority of the candidature, is an essential precondition for statements such as $3ab < -9b$ which with (*) leads immediately to the final inequality.

Question 6

This question generated a lot of good work. The majority of candidates produced a complete and [illegible] response.

Most candidates attempted to establish the required recurrence formula by using the method suggested rather than by using the integration by parts rule. After differentiation it is necessary to use the identity $x^2 \equiv (x^2+1)-1$ in some constructive way. However, a significant minority of candidates were unable to effect this and such a failure, which was very Centre dependent, indicates a lack of facility with basic algebra. Also, at the integration stage it was common for no limits or wrong limits to appear and this carelessness undermined otherwise correct strategies.

(ii) The foundations of a full proof are the results $2I_2 = \frac{\pi}{4} + \frac{1}{2}$ and $4I_3 = 3I_2 + \frac{1}{4}$ which follow immediately by taking n = 1, 2 in the reduction formula. Beyond that it is only necessary to quote or prove that $I_1 = \frac{\pi}{4}$ and then effect some simple algebra to obtain the required result. Most candidates argued correctly in this way but there were those who assumed the reduction formula to be true for $n = 0$ even though the question makes it clear that $n \geqslant 1$. No doubt this strategy was implemented so as to circumnavigate the need to evaluate I_1 but, in fact, it leads to an incorrect result.

Question 7

There were relatively few completely correct responses to this question and this outcome, most of all, was due to technical errors.

Almost all responses showed a correct answer to the first part of this question. Subsequently, the majority produced a broad strategy which was fundamentally correct. In particular, the obtaining of a real form for the denominator of the result for $\sum_{n=1}^{10} 2^{-n} e^{n\pi i/10}$ was usually effected accurately. It was in the final stage, where it is necessary to extract the imaginary part of the numerator, that solutions ran into confusion. Few candidates made obvious simplifications as their working developed, e.g., $z = e^{i\pi/10} \Rightarrow z^{10} = e^{i\pi} = -1$ and so arrived at their destination, if at all, only after a lot of unnecessary labour. In contrast, a small minority of candidates produced impressive working to prove what was required with a remarkable economy of effort.

Answer : $\dfrac{z\left(1-\left(\frac{z}{2}\right)^N\right)}{2-z}$

Question 8

The first part of this question was answered well by the majority of candidates. The simple ideas involved in the concluding part, however, eluded the majority.

Generally the correct integral representations of the coordinates of the centre of gravity of the first region were in evidence and these were usually evaluated correctly. Persistent errors were incorrect limits and in some cases no numerical limits at all, and the omission of the factor of $\frac{1}{2}$ in the formula for $\bar{y}$.

In the remainder of this question, there were many attempts to obtain the coordinates of the centroid of the second region by starting all over again. However, the question asked for deduction from the first results and only a minority of responses produced the required numbers in this way. Some candidates drew diagrams to help them on their way but nevertheless failed to produce a sensible result for $\bar{x}$ though usually, in such cases, the value of $\bar{y}$ was correct.

Answers: $\left(\frac{3}{5}, \frac{2}{35}\right)$, $\left(\frac{2}{5}, \frac{2}{35}\right)$.

Question 9

Correct overall strategies usually appeared in response to the first part of this question, but implementation was frequently undermined by elementary errors. Responses to the second part showed many conceptual errors so indicating a lack of understanding of the geometry. Overall, an expeditious solution requires at least three applications of the vector product and lack of accuracy in these evaluations seriously depressed marks. The determination of the shortest distance between the line *l* and the line, *m*, of intersection of the planes Π_1 and Π_2 was perceived by most to be the shortest distance of the given point, *P*, with position vector 4**i** + 5**j** + 6**k** to *m*. Thus they continued by evaluating the vector product (4**i**+ 5**j** + 6**k**) x (3**i** + **j** – **k**) to obtain –11**i** + 22**j** – 11**k**. The required minimum distance is then given immediately by $p = \dfrac{|-11\mathbf{i}+22\mathbf{j}-11\mathbf{k}|}{\sqrt{11}} = \sqrt{66}$. Alternatively *p* may be determined by the use of an orthogonality condition such as, for example, $3(4-3\lambda)+(5-\lambda)-(6+\lambda)=0$ which leads to $\lambda = 1$ and hence to $p = \sqrt{66}$. A small minority of candidates used this strategy with success.

Yet another strategy employed by some consisted first of all in the determination of the length of the projection of the line segment *OP* on *m*. This turns out to be $\sqrt{11}$. Application of Pythagoras' theorem then leads to $p = \sqrt{77-11} = \sqrt{66}$.

Answers: **r** = 4**i** + 5**j** + 6**k** + λ (3**i** + **j** – **k**); $\sqrt{66}$.

Question 10

Most candidates had, at least, some idea as to relevant methods, but again elementary working errors did a lot of damage to responses.

Correct eigenvectors appeared in almost all scripts. However, some candidates wasted examination time by first deriving the eigenvalues, even though they were given in the question. Subsequently, results for **D** and **P** were obtained, which almost always were consistent with the candidate's eigenvectors.

The majority of candidates used elementary row and column operations to obtain $\mathbf{P}^{-1}$ but much of this working was undermined by basic arithmetic errors. Very few used the cofactor method. In contrast, a significant minority showed no working at all and so, in accordance with the rubric, gained no credit.

In the final part of this question, most candidates inferred that $4^{-n}\mathrm{M}^n = \mathbf{P}\ \mathrm{diag}\left(\left(\frac{1}{4}\right)^n, \left(\frac{3}{4}\right)^n, 1\right)\mathbf{P}^{-1}$.

Subsequently, however, many responses went on to multiply out before taking the limit. This led to some very unwieldy and scarcely readable working. It is, of course, much easier to take the limit first. This leads to the argument:

$$\lim_{n\to\infty} 4^{-n}\mathbf{M}^n = \begin{pmatrix} 1 & 1 & 1 \\ -4 & -2 & -4 \\ -1 & -1 & -4 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{2}{3} & -\frac{1}{2} & \frac{1}{3} \\ 2 & \frac{1}{2} & 0 \\ -\frac{1}{3} & 0 & -\frac{1}{3} \end{pmatrix}$$

$$= \ldots = \begin{pmatrix} -\frac{1}{3} & 0 & -\frac{1}{3} \\ \frac{4}{3} & 0 & \frac{4}{3} \\ \frac{4}{3} & 0 & \frac{4}{3} \end{pmatrix}$$

Those candidates who argued in this way generally had little difficulty in producing complete and correct working.

Answers: $\begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -4 \\ -4 \end{pmatrix}$.

$\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ -4 & -2 & -4 \\ -1 & -1 & -4 \end{pmatrix}$, **D** = diag (1, 3^n, 4^n).

$$\mathbf{P}^{-1} = \begin{pmatrix} -\frac{2}{3} & -\frac{1}{2} & \frac{1}{3} \\ 2 & \frac{1}{2} & 0 \\ -\frac{1}{3} & 0 & -\frac{1}{3} \end{pmatrix}.$$

Question 11

Most candidates began by reducing **A** to an echelon form such as $\mathbf{E} = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ from which they deduced correctly the rank of **A**.

The better candidates then deduced that the dimension of the null space is 1 and hence, using an equation of the form **Ex** = **0**, easily obtained a result for the vector **e**. They also obtained a correct result for $\mathbf{x}_0$ in a systematic and clearly intelligible way from the given equation (*). Candidates who did less well with this question generally had more difficulty in obtaining **e** than $\mathbf{x}_0$. Their method did not involve a separate consideration of **Ex** = **0** but instead, they generally worked with the 4 linear equations represented by (*) in a haphazard way.

The final part of this question showed up a general deficiency in the understanding of inequality arguments, as was the case in **Question 5 (iii)**. The majority of candidates started from the general solution $\mathbf{x} = \begin{pmatrix} 1-2\lambda \\ 1-11\lambda \\ -1+5\lambda \\ \lambda \end{pmatrix}$, or, at least, something like it. Many subsequent arguments were either incomplete or erroneous. Thus some considered only $\lambda > 0$ and $\lambda < 0$ but not $\lambda = 0$. Others got as far as considering the two key inequalities $1-11\lambda > 0$, $-1+5\lambda > 0$ but still were unable to complete a convincing argument. In fact, these 2 inequalities taken together imply that $\lambda < \frac{1}{11}$ and $\lambda > \frac{1}{5}$ which is clearly impossible.

Answers: 3; $\mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{e} = \begin{pmatrix} -2 \\ -11 \\ 5 \\ 1 \end{pmatrix}$

Question 12 EITHER

The majority of responses showed correct derivations of the introductory, displayed results. The working proved to be increasingly less productive as the question developed.

Most responses showed that the result for S can be obtained by replacing N by $2N + 1$ in the sum of squares formula displayed in the question.

Most responses worked from $T = S - \sum_{n=1}^{N} (2n)^2$ or from $T = \sum_{n=1}^{N+1} (2n-1)^2$ and showed, at least, a correct unsimplified result. Beyond that there were many working errors to be seen so that many attempts did not lead to a correct properly factored result.

Few used the simple identity $U = 2T - S$ or even $U = S - 2\sum_{n=1}^{N} (2r)^2$ in an attempt to produce a result for U. Instead, most responses evaluated summation of positive terms and summation of negative terms without reference to previous results.

(i) Most of those who had obtained correct answers for S and T went on to establish correctly the behaviour of $\frac{S}{T}$ as $N \to \infty$. Others who had unsimplified, but correct, forms for these sums soon became involved in algebraic complexity which they were unable to resolve.

(ii) This final part can be answered by observing that $\frac{S}{U} = \frac{4N}{3} + 1$ is an integer if and only if N is divisible by 3, and that $\frac{T}{U} = \frac{2N}{3} + 1$ is an integer if and only if N is divisible by 3.

Answers: $S = \frac{1}{3}(N+1)(2N+1)(4N+3)$; $T = \frac{1}{3}(N+1)(2N+1)(2N+3)$; $U = (N+1)(2N+1)$;

(i) $\lim_{N\to\infty} \left(\frac{S}{T}\right) = 2$.

Question 12 OR

Most responses showed some progress with parts **(i)** and **(ii)**, but many errors in part **(iii)**.

(i) Nearly all responses showed the given point A to be on both C_1 and C_2. For B, however, arguments were hazy or non-existent and there few attempts to use the fact that $\cos\theta$ is an even function of θ.

(ii) About half of all sketches were complete in every essential respect. Some candidates appeared not to comprehend that C_1 is a circle. The restriction $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$ was ignored in some scripts.

(iii) Graphs drawn in part **(ii)** which were erroneous in some material way inevitably led to incorrect integral representations of the area, A, of the defined region. In fact, what should have appeared is $A = \int_0^{\alpha} (1+\cos\theta)^2 \,d\theta + 16\int_{\alpha}^{\pi/2} \cos^2\theta \,d\theta$.

Beyond that, use of $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ together with accurate working will lead to the required result.

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Paper 2

General comments

As in previous years, virtually all candidates attempted all the eleven required questions, and excellent work was produced by some candidates in all questions. While it was more common for candidates to perform somewhat better on the compulsory Statistics questions (**Questions 6-10)** than on the Mechanics ones (**Questions 1-5**), this was by no means universal. Likewise the Statistics alternative in **Question 11** was chosen more often than the Mechanics one, and in general produced rather better answers, but good answers were quite often seen to the first two parts of the Mechanics alternative. With the possible exception of **Question 6**, all the questions produced some challenges to all but the best-prepared candidates, as detailed below.

Comments on specific questions

Question 1

Although many candidates began by resolving forces horizontally and vertically, probably the quickest way to find the tension is to take moments about A. However those who did so very often included the moment of the horizontal component of the tension, but omitted the vertical one. A less common fault was to assume the reaction acts at the mid-point of AB, whereas it must act at A if the crate is on the point of lifting off the ground. The horizontal and vertical resolutions produce the friction and reaction respectively, and the least possible coefficient of friction follows from their ratio.

Answers: **(i)** 4470 N; **(ii)** 0.499.

Question 2

In the collision between the balls, the resulting velocity of either one is readily found by formulating and then solving the conservation of momentum and the restitution equations, and the impulse then follows. Few candidates encountered any difficulties with this, apart from arithmetical errors. The treatment of the second collision essentially depends on the component of the ball Q's velocity along the barrier being unchanged, while the normal component changes by a factor 0.7, the coefficient of restitution. A surprising number of candidates had this factor on the wrong side of their restitution equation, despite having handled it correctly in the first collision. Combination of the two components leads to $\tan^2 \alpha = \frac{1}{0.7}$ and hence α.

Answers: $\frac{57}{40}$; 50.1°.

Question 3

Finding the acceleration correctly requires an appreciation that the tensions T_A and T_B on either side of the wheel are unequal, and that the net force on each bucket is the difference between the relevant tension and the total weight of the bucket, plus contents in the case of B. The many candidates who did not appreciate this produced a much simpler, and invalid, model, in some cases taking no account of the motion of the wheel. The correct solution not only equates the product of the mass and acceleration of each bucket to the net force acting on it, but also equates the product of the moment of inertia $6mr^2$ of the wheel and its angular acceleration $\frac{d\omega}{dt}$ to $(T_B - T_A)r$. Combination of these three equations gives the acceleration $r\frac{d\omega}{dt}$ of B. Provided the first part has been answered correctly, the greatest possible value of λ is found easily by equating $T_B - T_A$ to mg.

Answers: $\frac{\lambda g}{8+\lambda}$; $\frac{8}{5}$.

Question 4

Although the details of the method of solution can vary, the usual underlying technique is to apply the standard SHM formula $v^2 = \omega^2(a^2 - x^2)$ at the first two tabular points, noting that the third point must be at a distance from the unknown centre of motion equal to the unknown amplitude a. Combining these three results yields the position of the centre of motion, as well as ω for use in the second part. The solution of this utilises another standard SHM result such as $x = a \cos \omega t$, applied at the first two tabular points. Several candidates wrongly used numerical values of the inverse trigonometric functions in terms of degrees rather than radians.

Answers: 2 cm left of the edge; 0.114 s.

Question 5

The key point here is that the radial force R between the skateboarder and the track must remain non-zero over the curved section. While most candidates rightly related R to the radial component of the skateboarder's weight and to the centripetal force, some did this only at the point B or less frequently at the mid-point of the curved track, whereas the critical point is C. The remaining challenge is to express the velocity at C in terms of x, and this is done by conservation of energy between A and C. Minor errors of accuracy, over angles for example, were common.

Answer: 7.57.

Question 6

This question seemed to attract more correct solutions than any other. Having calculated the nine expected values, the value 4.58 of χ^2 is found and compared with the tabular value 7.779, leading to the conclusion of independence.

Question 7

The correct method was usually employed for the required confidence interval, with only a few candidates using a biased estimate of the population variance instead of an unbiased one. More common was the use of wrong tabular t-value, which should be 2.998. A somewhat similar mistake was also seen in the second part, where the appropriate value is 2.326 since the sample is said to be large. Other occasional errors were to effectively take the required interval width to be 10 mm, or to misread the question as reducing the width *by* 5 mm instead of *to* 5 mm.

Answers: [117, 144]; 136.

Question 8

The assumption expected by the Examiners was that the two populations have the same variance, but this was often not stated. Rather fewer candidates gave incorrect hypotheses, in some cases relating the difference between the population means to zero rather than 20 cm. The unbiased estimate s of the common population variance may be calculated from $\frac{5{s_A}^2 + 3{s_B}^2}{8}$, but many candidates used instead factors 6 and 4 corresponding to biased estimates, or else entirely the wrong test corresponding to known population variances. The correct test requires the calculation of $\frac{165 - 136 - 20}{s\sqrt{\frac{1}{6} + \frac{1}{4}}}$, giving 1.50, and then its comparison with the critical value 1.86, leading to the conclusion that the horse seller's claim is not justified.

Question 9

The first part of this question was rarely answered in full. Consideration of the formulae for the me …, x_n and y_1, …, y_n shows that they are unaltered when an additional value equal to the mean is ad each set, and a reasonable number of candidates got this far. Some, however, simply stated that the m are unaltered, without demonstrating it, while others took the new mean to be the average of the origin mean and the additional value. If the regression line coefficient b is expressed in the form given in the *List of Formulae* then the additional terms in the summations are clearly zero, showing that b is unaltered when the extra pupil's marks are added. Those candidates who used a different formula for b often became involved in lengthy and ultimately unproductive algebraic manipulation, however. Finally it follows that the form of the regression line given in the *List of Formulae* is unaltered. Some candidates thought it was so obvious that the two regressions lines are identical that no proof was necessary, while others simply stated that a regression line passes through the point defined by the two means but without showing, for example, that the slope is unaltered by the addition of this point to the data. Another invalid approach was to use only the given data instead of considering the general case. By contrast almost all candidates found the required regression line for the given data correctly, and then calculated the value of y corresponding to $x = 46$. A variety of comments on the suitability of the method are possible, including the small size of the sample, the fact that 46 lies within the range of given values of x, and the observed or calculated lack of correlation.

Answers: $y = 45.8 - 0.271x$; 33.4.

Question 10

Most candidates substituted the correct Poisson terms into the given equation in order to find μ, though a few misread the factor 5 as 3. However a larger number produced a constant value for the probability that no train leaves in a randomly chosen t-hour period, without apparently questioning why their result was independent of t. Derivation of the cumulative distribution function of T, by subtracting the previous result from unity, eluded many candidates, some of whom instead wrote down what they hoped might be an appropriate exponential distribution involving μ. The final two parts follow respectively from evaluating the distribution function with $t = \frac{1}{6}$, and from equating it to $\frac{1}{2}$.

Answers: 10; e^{-10t}; $1 - e^{-10t}$; 0.811; 4.16 minutes.

Question 11 EITHER

Finding the moment of inertia I of the body in the first part requires use of the formula for the moment of inertia of a rectangular lamina about its centre, applied to both $ABCD$ and $PQRS$, and application of the parallel axes theorem to effectively transfer the moments of inertia to the axis through K. Although this alternative was not as popular as the Statistics one which follows it, many of the candidates who chose to attempt it handled this first part reasonably well, and knew the principle to be applied in the second part. This involves equating the loss $12mga$ in potential energy to $\frac{1}{2}I\omega^2$, yielding the given value of ω^2. Curiously many of these same candidates used $\frac{1}{2}mv^2$ instead of the formula appropriate to rotational motion for the kinetic energy in the final part, no doubt because the question spoke of the speed of the vertex C.

Answers: $\frac{40}{3}ma^2$; **(ii)** $6\sqrt{(ga)}$.

Question 11 OR

Provided the correct parameter $p = 0.596$ was found correctly, almost all candidates knew how to calculate the expected frequencies, and most rightly combined the first two cells before calculating the value 3.98 of χ^2. Comparison with the critical value 7.815 (if the two cells were combined) leads to the conclusion that the binomial distribution fits the data. Finding the value of x in the second part involves estimating μ from $100p$ and σ from $\sqrt{100p(1 - p)}$, and then equating $\frac{x - \mu}{\sigma}$ to 1.282, possibly with a continuity correction although the question only required the probability to be approximately 0.1. Some candidates made no attempt to find p at the beginning of the question, and instead simply took it to be $\frac{1}{2}$.

Answer: 66.