FURTHER MATHEMATICS

Paper 9231/11
Paper 11

General Comments

The scripts for this paper were of a generally good quality. There was a substantial number of high quality scripts and many showing evidence of sound learning. Work was well presented by the majority of candidates. Solutions were generally set out in a clear and logical order. The standard of numerical accuracy was good. Algebraic manipulation, where required, was of a high standard.

There was no evidence to suggest that candidates had any difficulty completing the paper in the time allowed. A very high proportion of scripts had substantial attempts at all twelve questions. Once again there were few misreads and few rubric infringements.

Candidates displayed a sound knowledge of most topics on the syllabus. Candidates tackled the questions on differential equations, polar co-ordinates, arc length, surface area of revolution, eigenvalues and eigenvectors confidently. The questions on proof by induction, complex numbers and reduction formulae proved to be challenging.

Comments on specific questions

Question 1

Most candidates were able to write down some, or, in most cases, all of the basic three equations for the relationships between the roots of a cubic equation. Many were then able to deduce, from them, the required result.

Question 2

Virtually all candidates were able answer the first request. Weaker candidates did not see how it could be used to form a relevant equation, using the method of differences, together with the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$. Where this was achieved, a common error was to think that $\sum_{r=1}^{n} 1$ was 1, rather than n. A substantial number of candidates, however, were able to gain full marks on this question.

Answer: $4r^3 + 6r^2 + 4r + 1$.

Question 3

A disappointingly large number of candidates thought that the first 'non-negative integer' was 1, rather than 0, so were unable to gain full marks on this question. Other candidates were not explicit enough in stating the inductive hypothesis. There were essentially two methods of showing that $\mathrm{H}_k \Rightarrow \mathrm{H}_{k+1}$. Taking $\mathrm{f}(k) = 11^{2k} + 25^k + 22$, one method was to consider f(k + 1) – f(k) and show that it was divisible by 24; the other method was to find a rearrangement of f(k +1) thus, having assumed that f(k) = 24λ for some non-negative integer λ, the result followed. Both methods were well used, with the first being somewhat more popular.

Question 4

This question was extremely well done. The rare error usually involved a small slip in solving the auxiliary equation, or the pair of simultaneous equations for the particular integral, resulting in the loss of an accuracy mark, but not method marks.

Answer: $x = e^{3t}(A\cos 4t + B\sin 4t) + 7\sin 2t + 4\cos 2t$.

Question 5

The vast majority of candidates were able to produce a sketch of a cardioid. Some lost a mark as there was no visible evidence of any sort of scale, either on the sketch itself or in the form of a table. Answers obtained solely from the use of a graphical calculator are penalised in this paper. The method of determining the area of a sector was well known. Most candidates used the appropriate double angle formula to deal with the integration of the $\sin^2\theta$ term. There were occasional slips in the final arithmetic, but many completely correct solutions.

Answer: $a^2\left(\frac{1}{4}\pi + 1 + \frac{1}{8}\sqrt{3}\right)$ (ACF).

Question 6

Candidates were well versed in the techniques required to do this question. Nevertheless, a small minority managed to find that the rank of **M** was 3. In this case they only had one vector in their basis for the null space of T, which scored a maximum of 2 marks in the second part of the question, if it was a correct component of the null space.

Answers: 2, any two independent column vectors from matrix **M**; any two of $\begin{pmatrix}-5\\-4\\0\\2\end{pmatrix}, \begin{pmatrix}-5\\0\\4\\2\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}$ (OE).

Question 7

In the first part of the question, most candidates made a reasonable attempt to derive the formula for $\tan 5\theta$, using the binomial theorem and de Moivre's theorem. As the answer was displayed, full working was expected, which caused the loss of a mark for some candidates who jumped to the result prematurely. In the middle part of the question, it was necessary to derive the roots from putting $\tan 5\theta = 0$ and explaining why $\tan\theta = 0$ was rejected, omission of which lost a further mark. The final part was usually done well, using the product of the roots of the equation. Those who did it from the information in the question, rather than from their previous work, were awarded one of the two marks as a special case.

Question 8

This question was done extremely well and there were many completely correct answers. The weakest candidates often made a very good attempt at this topic. It was pleasing to see that many candidates were remembering the key step in part **(ii)**: $\int 2\pi y\,\mathrm{d}s = \int 2\pi y \frac{\mathrm{d}s}{\mathrm{d}t}\,\mathrm{d}t$. which was evidence of good teaching. This was a feature that had caused problems in the past.

Answers: $1\frac{1}{3}$ or 1.33 (3 sf), $\frac{11}{9}\pi$ or 3.84 (3 sf).

Question 9

The first part of the question was well done, using the result **Me** = λ**e**, where **e** is an eigenvector and λ the corresponding eigenvalue. The vast majority used the given information to form two simultaneous equations, which they solved, in order to find the other two eigenvalues. The small minority who did not do what was asked in the question, but found the eigenvalues from the characteristic equation, received two of the four marks available. Good attempts were made, in the last part of the question, at finding corresponding eigenvectors. This question proved to be a good source of marks for a large number of candidates.

Answers: -1; -3 and -4; $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ (OE) and $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ (OE) respectively.

Question 10

The derivation of the reduction formula caused considerable difficulty for all but the best candidates. The problem was that few tried to write down anything for I_{n+1}. Those who did so usually managed to see that, by replacing $\sin^2 x$ by $1-\cos^2 x$ in the numerator, an integral was obtained which could lead to the result, using integration by parts. The calculation using the reduction formula was usually done well, even when unsuccessful attempts were made at the first part of the question. However, lack of complete working sometimes forfeited a mark, since the answer was a displayed result.

Question 11

Attempts at the first part showed that the vast majority had been taught to find the vector product of the direction vectors of the lines *AB* and *CD*. The shortest distance between the lines l_1 and l_2 could then be found, by finding the scalar product between a vector which joined any point on l_1 to any point on l_2, with the unit normal vector, previously found by the vector product. A small minority of candidates found a vector between general points on l_1 and l_2 then obtained scalar products between this vector and each of the direction vectors of l_1 and l_2, equated the scalar products to zero, thus forming a pair of simultaneous equations. These candidates were invariably defeated by the algebra involved. The second part of the question was well done. Most were able to use the vector product to find the normal to planes Π_1 and Π_2 and thus find the angle between the planes by use of the scalar product.

Answers: $\frac{16}{3}$ or 5.33 (3 sf); 36.7°.

Question 12

Either

A good number of candidates chose to tackle this question, but, sadly, it proved to be a poor choice for them. In part **(i)** finding $\frac{dy}{dx}$ was well done, but then many only found $\frac{d}{dt}\left(\frac{dy}{dx}\right)$, thinking that they had found $\frac{d^2y}{dx^2}$. A similar problem arose in part **(ii)** for, although candidates realised that they needed to find $\frac{1}{4}\int_0^4 y\,dx$, in converting the integral in terms of *x* into one in terms *of t*, the $\frac{dx}{dt}$ term was omitted. Some of these candidates met a little more success in part **(iii)**, since $\frac{1}{2}\int y^2 dx$ could be found using *x* rather than *t*. However, in order to obtain the final answer, $\int_0^4 y\,dx$ needed to be found and this was seldom done.

Answers: $\frac{1}{16t^3(2-t)^{\frac{3}{2}}}(4-3t)$ (ACF, but simplified); $\frac{8}{15}\sqrt{2}$ or 0.754; $\frac{5}{16}\sqrt{2}$ or 0.442.

Or

By contrast, those choosing this alternative met with greater success. It was straightforward to spot the value of d from the equation of the vertical asymptote. In part **(ii)**, by putting $x = 0$ in the equation of C the value of c was also easily found. By rearranging the equation of C to the form $y = ax + (b + 2a) + \frac{4 + 4a + 2b}{x - 2}$ and comparing with the equation of the oblique asymptote, the values of a and b were determined. Many candidates were thus able to acquire the first 7 marks in this question. The most efficient way of doing part **(iii)** was to rearrange the equation of C into a quadratic equation in x then use the discriminant to determine the range of values that y could take, for real values of x. It was good to see a considerable number of candidates adopting this approach. Clearly it was possible to use a method relying on differentiation, in which case it was <u>far</u> easier to write the equation of C in the form $y = x + 1 + 6(x + 1)^{-1}$ before differentiating twice. Using differentiation required a complete method. Frequently, candidates were unable to find the values of y corresponding to the values of x, where the first derivative was zero; they were also unable to find the value of the second derivative at these points.

Answers: **(i)** –2; **(ii)** 1, –1, 4.

FURTHER MATHEMATICS

Paper 9231/12
Paper 12

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The scripts for this paper were of a generally good quality. There was a substantial number of high quality scripts and many showing evidence of sound learning. Work was well presented by the majority of candidates. Solutions were generally set out in a clear and logical order. The standard of numerical accuracy was good. Algebraic manipulation, where required, was of a high standard.

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Answers: **(i)** –2; **(ii)** 1, –1, 4.

FURTHER MATHEMATICS

Paper 9231/13
Paper 13

General Comments

The scripts for this paper were of a generally good quality. There was a substantial number of high quality scripts and many showing evidence of sound learning. Work was well presented by the majority of candidates. Solutions were generally set out in a clear and logical order. This year saw a pleasing improvement upon last year's work. There were very few poor scripts. The standard of numerical accuracy was good. Algebraic manipulation, where required, was of a high standard.

There was no evidence to suggest that candidates had any difficulty completing the paper in the time allowed. A very high proportion of scripts had substantial attempts at all eleven questions. Once again there were few misreads and few rubric infringements.

Candidates displayed a sound knowledge of most topics on the syllabus. Candidates tackled both questions involving parametric equations well. There was also much good work on eigenvectors, differential equations and rational functions. The complex number question and the reduction formula question challenged a good many candidates.

Comments on specific questions

Question 1

There were several ways of going about the first request. The majority formed a 3×3 matrix ($\mathbf{M}$ say) from the column vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ and used elementary row operations to show that it had rank 3. Others found that Det$\mathbf{M}$ had the value $-4(\neq 0)$, while others showed that the only solution of $a\mathbf{a} + b\mathbf{b} + c\mathbf{c} = \mathbf{0}$ was $a = b = c = 0$. In any of these cases it was necessary to state that the work implied that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ were thus linearly independent, before saying that they formed a basis for $\mathbb{R}^3$. The second request was answered correctly by many. Too many candidates with the incorrect answer showed no working and thus lost both marks, whereas those with some sensible working, and making a small error, gained a method mark.

Answer: $\mathbf{d} = 2\mathbf{a} - \mathbf{b} + \mathbf{c}$.

Question 2

The first mark was lost by the minority of candidates who gave a specific numerical example, rather than the required algebraic proof. Most were then able to use either the initial hint, or partial fractions, to sum the given series to n terms. They then, in most cases, correctly deduced the sum to infinity of the series. The final mark could be earned on a follow through basis, if a small previous error had occurred.

Answers: $1 - \dfrac{1}{(n+1)^2}$ (OE), 1.

Question 3

Although a minority of candidates got full marks for this question on proof by induction, significant numbers lost marks through lack of rigour. Some failed to give a clear statement of the inductive hypothesis, that is, when P_k is true for some positive integer k, then $\phi(k) = 8\alpha$ for some integer α. Only the best candidates established $P_k \Rightarrow P_{k+1}$, and in most cases this was done by rearranging the expression for $\phi(k+1) - \phi(k)$ to obtain $5^k(16k + 24)$.

Question 4

This question on polar graphs was generally well done. The majority of candidates correctly obtained the given equation of *C*. A significant number of candidates produced a curve with four loops rather than two. The extra two loops corresponded to negative values of r, which is not allowed in the syllabus. Possibly, the majority of them were merely copying what came up on their graphical calculator. A common error in the final part of the question was to find the area of both loops, or to make some error with the limits of integration.

Answer: $\frac{1}{2}a^2$.

Question 5

Most candidates were able to state the sum of the series. The majority then realised that they needed to put $z = \cos\theta + i\sin\theta$ in the expression for the sum and then take the real part, in order to obtain the given result. Apart from an attempt to rationalise the denominator, many made little further progress. Better candidates obtained an expression such as $\dfrac{\cos\theta - \cos(n+1)\theta - 1 + \cos n\theta}{2 - 2\cos\theta}$, but only the best managed the trigonometrical manipulation required to obtain the displayed result.

Question 6

Both parts of this question were done extremely well by the vast majority of candidates. Weak candidates had difficulty in finding the square root of their expression for $\dot{x}^2 + \dot{y}^2$. Those who negotiated this hurdle invariably obtained the correct answer for part **(i)**. In part **(ii)** the integrals of $e^{\frac{3t}{2}}$ and $e^{\frac{t}{2}}$ sometimes produced expressions involving $e^{\frac{5t}{2}}$ and $e^{\frac{3t}{2}}$, despite candidates having integrated correctly in part **(i)**.

Answers: **(i)** $e^2 + 7$, **(ii)** $16\pi\left(\frac{2}{3}e^3 + 8e - \frac{26}{3}\right)$ (ACF).

Question 7

The majority of candidates were able to find $\dfrac{dy}{dx}$ correctly, but a substantial number only found $\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)$, rather than $\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) \times \dfrac{dt}{dx}$, when attempting to find $\dfrac{d^2y}{dx^2}$. In the second part, a common error was to find only one solution of the equation $\cos 2t = 0$, despite a plural being used in the request in the question.

Answers: $\dfrac{4\sin^3 t - 6\sin t}{\cos^3 t}$; $\left(\frac{1}{\sqrt{2}}, 1\right)$ (maximum), $\left(\frac{1}{\sqrt{2}}, -1\right)$ (minimum)

Question 8

The first piece of bookwork was only done well by the best candidates, who knew to pre-multiply both sides of the equation $\mathbf{Ae} = \lambda\mathbf{e}$ by $\mathbf{A}^{-1}$. The second piece of bookwork was done rather better, by a substantial number of candidates. Most candidates correctly found an eigenvector corresponding to the eigenvalue of 1. The equation $\mathbf{Ae} = \lambda\mathbf{e}$ was then used to find the eigenvalues corresponding to the given eigenvectors. Those who found the eigenvalues from the characteristic equation, but did not match them to the eigenvectors, lost marks. The matrix $\mathbf{P}$ was invariably written down correctly, and for the most part a correct diagonal matrix $\mathbf{D}$ was found. Occasionally arithmetical errors crept in, and sometimes candidates forgot to cube elements.

Answers: $\begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix}$ (OE); 2, 3; $\mathbf{P} = \begin{pmatrix} -1 & 0 & 1 \\ -4 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & \frac{125}{8} & 0 \\ 0 & 0 & \frac{1000}{27} \end{pmatrix}$.

Question 9

Sadly, many answers for the first request were left in terms of u rather than θ. The opening hint did not prove useful to many candidates who were unaware how to split the integrand, in order to integrate by parts. A good number of those who successfully negotiated this step then did not see the need to replace $\cos^2\theta$ by $1-\sin^2\theta$, hence few were able to correctly deduce correctly the reduction formula. Candidates were much more successful in finding I_0 and using the reduction formula to find I_4.

Answers: $-\frac{1}{3}\cos^3\theta + c$; $\frac{\pi}{32}$.

Question 10

This question was well done by a large number of candidates. Only the weakest forgot the correct form of the complementary function, when the auxiliary equation had repeated real roots. There was a significant minority of candidates working with variables (x,y) rather than (t,x). Those who obtained the general solution usually made a good attempt at finding the values of the arbitrary constants from the initial conditions. Many were correctly able to deduce that $\frac{\mathrm{d}x}{\mathrm{d}t} \to 50$ as $t \to \infty$.

Answer: $x = 100 + 50t - (100 + 58t)\mathrm{e}^{-0.08t}$.

Question 11

Either

This was by far the more popular alternative and the question was generally well done. Most candidates obtained correctly the partial fractions in the first part. They then went on to use this expression to show that $\frac{dy}{dx} = 0$ when $x^2 - 14x + 1 = 0$. The majority then used the discriminant to show that this quadratic equation had two real roots. Some, however, found the two roots, a method which was equally acceptable. The asymptotes were correctly identified by almost all, but only the best showed that the graph could not intersect

the x-axis, since $y = 0 \Rightarrow 2x^2 - x + 5 = 0$ which has no real roots, having a discriminant with a value of -39.

Lack of working here lost one mark. Although the majority found the intersection of C with the horizontal asymptote, only the best candidates deduced the correct shape for the right hand branch of C.

Answers: $A = 3,\ B = -4;\ x = 1,\ x = -1,\ y = 2,\ (7, 2)$.

Or

Although less popular, this question was well done by most of those attempting it and there was a good number of completely correct solutions. In part **(i)**, nearly all put $\lambda = \mu = 0$ in order to show that the point A was in the plane Π_1. Then, thinking of the equation of Π_1 in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{p} + \mu\mathbf{q}$, they found $\mathbf{p} \times \mathbf{q}$ in order to obtain the requisite normal vector in part **(ii).** The point L was found without difficulty and most candidates used it to find the cartesian equation of the plane Π_2 successfully. Only the best candidates used the fact that $\mathbf{n} = t(2\mathbf{i} - \mathbf{j} + 5\mathbf{k})$ to obtain $t = 2$ and hence the position vector of the point N. In the final part the majority, having found the point M, used the vector product of either $\overrightarrow{OM}$ or $\overrightarrow{NM}$ with $\hat{\mathbf{n}}$ to find the required perpendicular distance. A few candidates felt more comfortable using the scalar product and Pythagoras in order to achieve the same result. There were some more laborious methods employed in parts **(iii)** and **(iv)**, but overall those who chose this alternative made few arithmetical errors.

Answers: **(ii)** $2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ (OE), **(iii)** $4\mathbf{i} - 2\mathbf{j} + 10\mathbf{k}$, **(iv)** 8.16 (CAO).

FURTHER MATHEMATICS

Paper 9231/21
Paper 21

Key messages

To score full marks in the paper candidates must be proficient in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

All steps of the argument or derivation should be presented in those questions which require a solution to be proved rather than just found, often indicated by words such as “show that” or “justify your answer”. Although less detail may suffice in questions where an unknown value or result is to be found, candidates are advised to show sufficient working so that credit may be earned for their valid method of solution even if an error occurs during its execution.

Non-standard symbols introduced by candidates should be defined or explained, for example by showing forces or velocities in a diagram within their written answers. Annotating a diagram in the question paper is insufficient, since this will not be seen by the Examiners.

General comments

Most candidates attempted all the questions, and the paper discriminated well between different levels of ability. As usual all questions were answered well by some candidates, most frequently **Questions 1**, **9** and **10**. **Question 3** and the last part of **Question 7** were found to be challenging by some but by no means all candidates. In the only question which offered a choice, namely **Question 11**, there was a strong preference for the Statistics option on this occasion.

Advice to candidates in previous reports to set out their work clearly, with any corrections legible and the replacements to deleted attempts readily identifiable, was seemingly heeded by many, though not all. When relevant in Mechanics questions it is helpful to include diagrams which show, for example, what forces are acting and also their directions, as in **Question 11** and the directions of motion of particles, as in **Question 3**. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. Thus in **Questions 8** and **11** any symbols used for variables when stating the hypotheses should either be in standard notation or should be correctly defined. When stating the conclusion of a statistical test, candidates should relate the wording of their conclusion to the purpose of the test as posed in the question.

Many candidates appreciated the need to explain their working clearly in those questions which require certain given results to be shown to be true rather than finding unknown results. It is particularly advisable to explain or justify any new equations which are written down, for example by stating that forces are being resolved in a specified direction, moments are being taken about a specified point, or moments of inertia are being found about a particular axis. Even when an unknown result must be found, such as a numerical value in a Statistics question, candidates are well advised to show their method, since credit may then be earned for a valid approach if, for example, an incorrect result is due only to an arithmetical error.

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Most candidates successfully used the given impulse to find the speed $4u$ of the sphere P before it strikes the barrier, and then Newton's law of restitution to find its subsequent speed, though occasionally the mass of P was wrongly taken as m rather than $2m$.

Answer: $3u$.

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By applying the standard SHM equation $v^2 = \omega^2(a^2 - x^2)$ to the particle P's speed at both A and B, the given ratio of its kinetic energy which of course equals that of the square of its speed at the two points yields an equation for a^2 and hence the required amplitude a. Not all candidates understood the precise meaning of the given information about the ratio, however, and sometimes took the kinetic energy at one point as a fraction such as 12/(11+12) of that at the other point. Having found a, the value 0·3 of ω can be found immediately from the SHM formula quoted above since v takes its given maximum value 0·6 when $x = 0$. The times at A and B can be found by expressing x as $a\sin \omega t$ or $a\cos \omega t$ with x respectively 0·5 and 0·75, but candidates need to consider carefully exactly what times their chosen expression represents when combining their times at A and B in order to find the required time for P to travel from A to B.

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Answer: $e = \frac{2}{3}$; $k = 2\frac{1}{4}$.

Question 4

The reaction R at the lowest point follows from equating $R - mg$ to the product of the mass m and the radial acceleration v^2/a, with the speed v at this point found by conservation of energy from the initial point A. A very similar application of conservation of energy produces the speed V_B of the bead as it leaves the wire at B, but candidates should realise that it is the vertical component of this speed, namely $v_B = V_B \cos \theta$, which determines the greatest height $v_B^2/2g = \frac{3}{8}a$ above B. This point B is of course $a\sin \theta = \frac{1}{4}a$ below the level of O. Those candidates who quoted an equation based on the motion of a projectile did not always realise that the angle θ in the standard form of this equation is not the same as the angle denoted by θ in this question.

Answers: (**i**) 3·3 mg; (**ii**) $2\sqrt{(ga/5)}$.

Question 5

As usual in such questions there is no unique method of solution, though most candidates found the individual moments of inertia of the five constituent parts about the axis through A using the parallel axes theorem where necessary, and summed these to yield the given moment of inertia I. Since the centre of mass of the composite object is at the centre of the rectangle, it is also legitimate to find the total moment of inertia of the two pairs of opposing strips and the rectangular sheet about an axis through this centre, and then apply the parallel axes theorem once to find I. As in all such questions where a given result must be shown, candidates should include sufficient details of their solution so as to justify full credit. A common failing was to include a moment of inertia without specifying an axis, or to use the same symbol for different moments of inertia. Candidates who simply write down a sum of terms with no explanation whatever run considerable risk, since an error in only one term can cast doubt on the validity of their whole process and thereby lose considerable credit. To some extent this is true in the final part where writing down only an incorrect equation without explanation may earn little credit. It is therefore sensible to state, for example, that the centre of mass of the object is at the mid-point of AC before writing down an equation of circular motion involving $\sin\theta$, where θ is the angular displacement at time t, and approximating $\sin\theta$ by θ to derive the standard form of the SHM equation in θ.

Answer: $(16\pi/7)\sqrt{(6a/g)}$.

Question 6

The expected value is $1/p$ and the two required probabilities are simply q^3p and $1 - q^5$ with $q = 1 - p$ and $p = ¼$, but a relatively large proportion of candidates took instead $p = ½$.

Answers: 4; (**i**) 27/256 or 0·105; (**ii**) 781/1024 or 0·763.

Question 7

Most candidates wrote down the probability density function correctly and many also found the required probability though some wrongly found its complement. The final part was more challenging, however, and incorrect variations on the essential inequality $(e^{-t/1000})^{10} \geq 0{\cdot}9$ were often seen. Once formulated correctly, its solution for t is straightforward.

Answers: (**i**) $e^{-t/1000}/1000$ $(t \geq 0)$, 0 $(t < 0)$; (**ii**) e^{-2} or 0·135; 10·5.

Question 8

After stating the hypotheses, which should be in terms of the population means and not the sample means, most candidates found unbiased estimates s_x^2 and s_y^2 of the population variances using X's and Y's sample respectively, and then an estimate s^2 of the population variance for the combined sample using $s^2 = s_x^2/60 + s_y^2/50$. The magnitude of z needed for the test is then $(101 - 95)/s$ which is approximately 1·96. Given the large size of the samples, it is acceptable here to use biased estimates instead, leading to 1·97. In either event, comparison with the tabular z-value of 2·326 leads to acceptance of the null hypothesis that the expenses claims from the two branches are the same on average. A minority of candidates made the assumption of equal population variances and used a pooled estimate of this common variance to produce a z-value of magnitude 1·97. Although not suggested by the question, the Examiners accepted this test provided the underlying assumption was stated explicitly.

Question 9

In general this question was answered well. The given value of p is found by integrating 200 f(x) over the interval (2, 3) while q may be found by similar integration over (4, 5) or by noting that the sum of the expected frequencies should equal 200. After stating the hypotheses to be tested, the calculated χ^2-value 11·3 is compared with the tabular value 12·6, leading to acceptance of the null hypothesis. Thus the specified distribution does indeed fit the observations.

Answer: q = 21·45 or 21·46.

Question 10

Most candidates used the usual formula to find the product moment correlation coefficient r for the samples and went on to state the null and alternative hypotheses, which should be in the form $\rho = 0$ and $\rho \neq 0$, though some wrongly stated them in terms of r which is not the same entity as ρ. Comparison of the calculated value of r with the tabular value 0·632 leads to a conclusion that there is evidence of non-zero correlation. Finding the gradient p from the given summations and hence the value of the constant q usually presented few problems for the many candidates who did not confuse the required equation with that of the regression line of y on x.

Answer: 0·687; $x = 1{\cdot}20y + 9{\cdot}92$.

Question 11 (Mechanics)

This optional question was much less popular than the Statistics alternative discussed below, and few complete attempts were seen, though the first part did not cause candidates too many problems. The length $4a$ of AB may be found by, for example, applying Pythagoras's Theorem to the triangle with hypotenuse AB formed by drawing a horizontal line from B to intersect AC. The angle CAB is then readily shown to be 60°, and hence the angle SAB is 30° as required. Even though the diagram suggests (correctly) that AS when extended passes through D, candidates should not assume this to be so without justification. As for the remaining parts of the question, the involvement of several unknown forces such as here usually offers candidates several methods of solution, and some preliminary thought as to the optimum choice is often beneficial. Thus resolving in the direction PQ will yield the normal reaction N_A at A immediately, since the other two unknown forces on the frame (the friction F_A at A and the reaction N_B at B) are not involved. While moments about A will give N_B, finding the contribution of W is non-trivial and so it may be preferable to use two easier independent resolution and moment equations and eliminate N_B from them to yield the required frictional force F_A. Such a deliberate strategy will serve candidates better than repeatedly resolving forces and taking moments at random in the possibly vain hope that a solution will somehow emerge.

Answer: **(iii)** $3W(2 - \sqrt{3})/8$.

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Most candidates attempting this popular option estimated the population variance s_P^2 correctly, and combined it with the appropriate t-value 2·365 to find the required confidence interval. A correct formulation of the essential inequality $(5{\cdot}35 - k) / \sqrt{(s_P^2/8)} \geq 1{\cdot}415$ in order to find the greatest possible value of k was not so common, however. The final part of the question specifies the use of a 2-sample test, and many candidates both complied with this requirement and conducted the test well. The unbiased estimate of the common population variance is 1·589. This gives a calculated value of t of 1·30, and comparison with the tabulated value 1·33 leads to the conclusion that the mean time for College Q is not less than for College P. As in all such tests, candidates should state the hypotheses explicitly in terms of the population means. The question specified that the necessary assumptions be given, but relatively few candidates stated that the time for College Q should, as for College P, have a normal distribution and that the two distributions should have the same variance.

Answer: (4·51, 6·19); 4·85.

FURTHER MATHEMATICS

Paper 9231/22

Paper 22

Key messages

To score full marks in the paper candidates must be proficient in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

All steps of the argument or derivation should be presented in those questions which require a solution to be proved rather than just found, often indicated by words such as "show that" or "justify your answer". Although less detail may suffice in questions where an unknown value or result is to be found, candidates are advised to show sufficient working so that credit may be earned for their valid method of solution even if an error occurs during its execution.

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Most candidates attempted all the questions, and the paper discriminated well between different levels of ability. As usual all questions were answered well by some candidates, most frequently **Questions 1**, **9** and **10**. **Question 3** and the last part of **Question 7** were found to be challenging by some but by no means all candidates. In the only question which offered a choice, namely **Question 11**, there was a strong preference for the Statistics option on this occasion.

Advice to candidates in previous reports to set out their work clearly, with any corrections legible and the replacements to deleted attempts readily identifiable, was seemingly heeded by many, though not all. When relevant in Mechanics questions it is helpful to include diagrams which show, for example, what forces are acting and also their directions, as in **Question 11** and the directions of motion of particles, as in **Question 3**. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. Thus in **Questions 8** and **11** any symbols used for variables when stating the hypotheses should either be in standard notation or should be correctly defined. When stating the conclusion of a statistical test, candidates should relate the wording of their conclusion to the purpose of the test as posed in the question.

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Answer: (4·51, 6·19); 4·85.

FURTHER MATHEMATICS

Paper 9231/23
Paper 23

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Comments on specific questions

Question 1

Two equations must first be formulated, based on conservation of momentum and Newton's restitution equation, and then solved for the speed of *A* after the collision. Since the question requires that this be shown to equal the given expression, candidates are advised to give some indication of their process of solution, rather than simply writing down the expression without explanation. Equating the given magnitude of the impulse to the change in momentum of either sphere yields a linear equation for *k*, which is readily solved. Although the reference in the question to the impulse experienced by *B* led many candidates to find the speed of *B* after the collision, this is of course unnecessary since the sphere *A* must experience the same impulse. Frequent causes for error when formulating the equation for *k* included incorrect signs and taking the mass of *B* as *m* rather than *km*.

Answer: $k = 2$.

Question 2

This question was found challenging by many candidates, often because of an apparent mistaken belief that the acceleration of *P* is directed towards the centre *O*, so in effect having no transverse component. The radial component is found by differentiating $\sin^2 2t$ to give the angular speed ω, and then forming $a\omega^2$. The transverse component requires a further differentiation to give $8 \cos 4t$, and equating this to zero gives a set of values of *t* from which the smallest positive one is selected. The magnitude of the resultant acceleration of *P* is found from the positive square root of the sum of the squares of the radial and transverse components with $t = \pi/12$, and hence the required resultant force. As always, candidates should read the question carefully, noting in this case that it is the resultant force that is asked for, not the acceleration.

Answers: (**i**) $\pi/8$ or 0·393; (**ii**) $5\ ma$.

Question 3

Three equations must be formulated, relating the impulse of magnitude *J* to the initial speed *u*, relating *u* to the speed *v* at angle θ using conservation of energy, and finally equating the net radial force to mv^2/a. The speeds *u* and *v* are then eliminated to yield the given expression for the tension *T*. As in other questions requiring that the validity of a given result be demonstrated, candidates should set out clearly every step of their argument if full credit is to be earned. In particular, in some attempts *J* only appeared at the very end when the term mu^2/a was replaced by J^2/ma without explicit justification. In part (**ii**), completely correct answers were rarely seen, partly but not solely because it is the motion of *P* and not the value of *T* which must be described, and because answers must be justified. A common response to the case $k = 1$ was to find that $T = 0$ when $\cos \theta = ⅓$, perhaps with some remark such as the string becomes slack, *P* does not complete a circle, or *P* then moves like a projectile. In fact, having received only a relatively small impulse, *P* cannot reach this point since $v = 0$ when $\cos \theta = ½$, and so *P* oscillates on either side of the downward vertical at *O*. Similarly when $k = 6$ candidates should consider both *v* and *T*, concluding that both remain positive for all values of θ, and so *P* continues to move indefinitely in a circle centred on *O*. This is the behaviour that might well be expected as a result of a relatively large impulse. Indeed in problems such as this, candidates may well benefit from some initial thought about the likely physical behaviour of the system before exploring it mathematically.

Question 4

Questions involving several unknown forces such as this usually offer candidates several methods of solution, and some preliminary thought about the optimum method is often beneficial. Noting in advance that the given inequality for *d* may probably be deduced from the normal reaction R_A at *A* being positive, and that the given inequality for μ will probably follow from $F_A \le \mu R_A$ where F_A is the friction at *A*, suggests that only R_A and F_A need be found. This can be achieved by resolving forces along *AB* and taking moments about *C*, or even better about the point at which the normal to the rod at *C* intersects the plane since this gives R_A immediately. A popular alternative is to resolve forces vertically and horizontally, though the resulting introduction of the unknown reaction R_C at *C* requires a third equation, most sensibly moments about *A* though others are possible. As it happens, the equations required by this approach are relatively straightforward, and so might be preferable despite the introduction of R_C. In any event such a deliberate strategy may well serve candidates better than repeatedly resolving forces and taking moments at random in the possibly vain hope that a solution will somehow emerge.

Answer: $(3/2 - 9d/25a)mg$.

Question 5

Once again there is no unique method of solution, though all involve finding the individual moments of inertia of the three constituent parts about appropriate axes, and then combining these with use of the parallel axes theorem where necessary to yield the given moment of inertia I. As in all such questions where a given result must be shown, candidates should include sufficient details of their solution so as to justify full credit. A common failing was to include a moment of inertia without specifying an axis, or to use the same symbol for different moments of inertia. Candidates who simply write down a sum of terms with no explanation whatever run considerable risk, since an error in only one term can cast doubt on the validity of their whole process and thereby lose considerable credit. To some extent this is true in the final part where writing down without explanation only an equation which does not yield the correct value of k may earn little credit. It is therefore sensible to state, for example, that the greatest angular speed ω occurs when C is vertically below the axis at A, and then to equate $\frac{1}{2}I\omega^2$ to the combined loss of potential energy of the two laminas and the particle.

Answer: $k = 0{\cdot}502$.

Question 6

The question specifies the use of a paired-sample t-test, and most candidates both complied with this requirement and conducted the test well. Use of a test other than the one specified is not acceptable, so candidates must be familiar with the names of all the tests included in the syllabus. The first step in the calculation is to find the differences between the pairs of observations and base the test on them. It is course essential to retain the signs of the differences and not just consider their magnitudes. The mean of the resulting sample is then 34/10 and the unbiased estimate of the population variance is 2662/45 or 59·16. This gives a calculated value of t of 1·398, and comparison with the tabulated value 1·383 leads to the conclusion that the population mean number of hours of absence has decreased. As in all such tests, candidates should state their hypotheses explicitly in terms of the population and not the sample means.

Question 7

All except the final part of this question were answered well by most candidates. The first three parts require the evaluation in turn of $(\frac{3}{4})^4 \times \frac{1}{4}$, $(\frac{3}{4})^8$ and $1 - (\frac{3}{4})^6$. Denoting this latter probability that James qualifies for the competition by P_J and the corresponding probability of Colin qualifying by P_C, the probability of exactly one of them qualifying may be found from $P_J(1 - P_C) + P_C(1 - P_J)$. As in other such probability questions, other methods of solution may be equally valid, though possibly more tedious.

Answers: (**i**) 0·079; (**ii**) 0·1; 0·822; 0·235.

Question 8

Most candidates produced good answers to this question. After stating the hypotheses to be tested, a table of the expected numbers of adults is produced in the usual way and preferably to an accuracy of two (or more) decimal places. The calculated χ^2-value 12·6 (or 12·7 if the expected numbers were found to only one decimal place) should be compared with the tabular value 9·488, leading to rejection of the null hypothesis. Thus the preferred type of car is dependent on age-group.

Question 9

While many candidates found the required value of k correctly by equating 0·6 to $1 - (k/8 - \frac{1}{4})$, some equated instead $k/8 - \frac{1}{4}$, and a small number effectively treated X as a discrete variable, taking only integer values, rather than a continuous one. When finding the distribution function of Y, its value should be stated for all values of y, and it should also be made clear on the graph of its probability density function that it is zero outside the central interval of y.

Answer: $k = 5{\cdot}2$; 0 ($y < \ln 4$), $e^{y/2}/8 - \frac{1}{4}$ ($\ln 4 \le y \le \ln 100$), 1 ($y > \ln 100$); $e^{y/2}/16$ ($\ln 4 \le y \le \ln 100$), 0 otherwise.

Question 10

As in all such tests, the hypotheses required in the first part should be stated in terms of the population mean and not the sample mean. The unbiased estimate 703/350 or 2·009 of the population variance may be used to calculate a *t*-value of 1·40. Since it is a one-tail test, comparison with the tabulated value of 1·415 leads to acceptance of the null hypothesis, namely that the population mean length is greater than 15·8 cm. The confidence interval is found in the usual way using a tabular *t*-value of 2·365, but incorrect alternatives were not uncommon.

Answer: (**ii**) (15·3, 17·7).

Question 11 (Mechanics)

This less popular option produced a variety of attempts. The given modulus of elasticity follows from equating the tension at the equilibrium point, at which the length of the string is $(12/7)l$, to the downward force mg. It may also be obtained from conservation of energy between the two points at which the particle is momentarily at rest. The force acting on P at a general point distant x from the equilibrium point, consisting of mg and the tension, should then be equated to $m\mathrm{d}^2x/\mathrm{d}t^2$, which yields the standard form of the SHM equation with ω^2 here $7g/5l$. The period then follows from $2\pi/\omega$. The time in the final part is found most easily by expressing the speed v of P in the form $v_0 \sin \omega t$, since it is then simply $\omega^{-1} \sin^{-1} ½$. However it may also be found in other, more lengthy, ways. If the displacement x is expressed as $x_0 \cos \omega t$ where $x_0 = l/7$ for example, then the value $(\sqrt{3}/14)l$ of x for which $v = ½\, v_0$ may first be determined using the standard SHM formula $v^2 = \omega^2 (x_0^2 - x^2)$, and then used to find t from $\omega^{-1} \cos^{-1} (x/x_0)$. Candidates should take care when deciding whether to use cosine or sine in such SHM formulae, basing their decision on the point from which time is measured.

Answers: (**ii**) $2\pi \sqrt{(5l/7g)}$; (**iii**) $(\pi/6) \sqrt{(5l/7g)}$.

Question 11 (Statistics)

The test requires an explicit statement of the null and alternative hypotheses, $\rho = 0$ and $\rho > 0$, and here candidates should be aware that r and ρ are not the same entity. Comparison of the given value 0·6 of r with the tabular value 0·497 leads to a conclusion that there is evidence of positive correlation. The required values of a and b are found by combining the given relation with the fact that the product ab of the gradients of the regression lines must equal $0{\cdot}6^2$ (not 0·6 as some candidates mistakenly stated). This leads to a quadratic equation in either a or b and the positive pair of gradients is chosen since r is given to be positive. The key to finding the value of c is that both regression lines are satisfied by the mean observed values of x and y. Not all candidates who found the regression line of x on y were able to sketch it on the same diagram as the other line, often effectively sketching instead $y = ax + c$. Finally many candidates appreciated that the coefficient of x in the equation of the regression line of z on x is equal to $5b$, and that the product moment correlation coefficient is unchanged by the scaling, though a justification for the latter result was not always attempted.

Answers: (**ii**) $a = 0{\cdot}4$, $b = 0{\cdot}9$; (**iii**) $c = 1{\cdot}72$; (**iv**) 4·5, 0·6.