ZNOTES.ORG

UPDATED TO 2020-22 SYLLABUS

CAIE IGCSE ADD MATHS (0606)

SUMMARIZED NOTES ON THE THEORY SYLLABUS

1. Functions

- **One-to-one functions**: each x value maps to one distinct y value (check using vertical line test)

e.g.

$$f(x) = 3x - 1$$

- **Many-to-one functions:** there are some $f(x)$ values which are generated by more than one x value

e.g.

$$f(x) = x^2 - 2x + 3$$

Domain =x values **Range** = y values

- **Notation:** $f(x)$ can also be written as $f : x \mapsto$
- **To find range:**
  - Complete the square

$x^2 - 2x + 3 \to (x-1)^2 + 2$

- Work out min/max point

Minimum point $= (1, 2)$
$\therefore$ all y values are greater than or equal to 2. $f(x) \geq 2$
One-to-many functions do not exist

- Domain of $g(x) =$ Range of $g^{-1}(x)$
- **Solving functions:**
  - $f(2)$: substitute $x = 2$ and solve for $f(x)$
  - $\mathrm{fg}(x)$: Substitute $x = g(x)$
  - $f^{-1}(x)$: let $y = f(x)$ and make x the subject
- **Composite Functions:**
  - $f(g(x))$ *or* $f \cdot g(x)$
  - Substitute all instances of x in f(x) with g(x)
  - Simplify
  - If it is $f^2(x)$, *or* $f(f(x))$, then for every x in f(x) substitute f(x)'s contents
- **Inverse Functions**
  - *Only* 1 *to* 1 *functions have inverses*
  - If f(x) is a function, equate f(x) to y
  - Replace all occurrences of x in f(x) with y
  - Try to make x the subject of the function again
  - That is the $f^{-1}(x)$
- **Transformation of graphs:**
  - $f(-x)$: reflection in the y-axis
  - $-f(x)$: reflection in the x-axis
  - $f(x) + a$: translation of a units parallel to y-axis
  - $f(x+a)$: translation of $-a$ units parallel to x-axis

2. Quadratic Functions

- **To sketch** $y = ax^2 + \mathrm{bx} + c$; $a \neq 0$
- **Determine the shape**
  - $a > 0$ – u-shaped $\therefore$ minimum point
  - $a < 0$ – n-shaped $\therefore$ maximum point
- **Use the turning point**
  Express $y = ax^2 + \mathrm{bx} + c$ as $y = a(x-h)^2 + k$ by completing the square

$$x^2 + nx \iff \left(x + \frac{n}{2}\right)^2 - \left(\frac{n}{2}\right)^2$$

$$a(x+n)^2 + k$$

$$\text{Where the vertex is } (-n, k)$$

- **Find the y-intercept:**
  - Substitute x as 0 to get y intercept
- **Find the x-intercept:**
  - Factorize or use formula
- **Type of root** by calculating discriminant $b^2 - 4ac$
  - If $b^2 - 4ac = 0$, real and equal roots
  - If $b^2 - 4ac > 0$, real and distinct roots
  - If $b^2 - 4ac < 0$, no real roots
- **Intersections of a line and a curve:** if the equations of the line and curve leads to a quadratic equation then:
  - If $b^2 - 4ac = 0$, line is tangent to the curve
  - If $b^2 - 4ac > 0$, line meets curve in two points
  - If $b^2 - 4ac < 0$, line does not meet curve
- Quadratic inequality:
  - $(x-d)(x-\beta) < 0 \Longrightarrow d < x < \beta$
  - $(x-d)(x-\beta) > 0 \Longrightarrow x < d \text{ or } x > \beta$

3. Equations, inequalities and graphs

- **Transformation of graphs:**
  - $f(-x)$: reflection in the y-axis
  - $-f(x)$: reflection in the x-axis
  - $f(x) + a$: translation of a units parallel to y-axis
  - $f(x+a)$: translation of $-a$ units parallel to x-axis
  - $f(ax)$: stretch, scale factor $\frac{1}{a}$ parallel to x-axis
  - $\mathrm{af}(x)$: stretch, scale factor a parallel to y-axis
- **Modulus function:**
  - Denoted by $|f(x)|$
  - Modulus of a number is its absolute value
  - Never goes below x-axis
  - Makes negative graph into positive by reflecting negative part into x-axis
- **Solving modulus function:**
  - Sketch graphs and find points of intersection
  - Square the equation and solve quadratic
- **Relationship of a function and its inverse:**
  - The graph of the inverse of a function is the reflection of a graph of the function in y = x

4. Indices & Surds

4.1. Indices

- **Definitions:**
  - for $a > 0$ and positive integers p and q

$$a^0 = 1 \qquad a^{-p} = \frac{1}{a^p}$$

$$a^{\frac{1}{p}} = \sqrt[p]{a} \qquad a^{\frac{p}{q}} = \left(\sqrt[q]{a}\right)^p$$

- **Rules:**
  - for $a > 0$, $b > 0$ and rational numbers m and n

$$a^m \times a^n = a^{m+n} \qquad a^n \times b^n = (\mathrm{ab})^n$$

$$\frac{a^m}{a^n} = a^{m-n} \qquad \frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$$

$$(a^m)^n = a^{\mathrm{mn}}$$

4.2. Surds

Definition
An irrational root is a surd, not all roots are surds

Rationalizing the Denominator
When the denominator is a surd, we can simplify by multiplying both the numerator and the denominator by the rationalization factor to rationalize

$$\frac{5}{3-\sqrt{2}} = \frac{5(3+\sqrt{2})}{(3-\sqrt{2})(3+\sqrt{2})} = \frac{15+5\sqrt{2}}{9+3\sqrt{2}-3\sqrt{2}-2} = \frac{15+5\sqrt{2}}{7}$$

5. Factors of Polynomials

- To find unknowns in a given identity
  - Substitute suitable values of x
    OR
  - Equalize the given coefficients of like powers of x

Factor Theorem:

- If $(x-t)$ is a factor of the function $p(x)$ then $p(t) = 0$

Remainder Theorem:

- If a function $f(x)$ is divided by $(x-t)$ then:

$$Remainder = f(t)$$

- The formula for remainder theorem:

$$Dividend = Divisor \times Quotient + Remainder$$

6. Simultaneous Equations

- Simultaneous linear equations can be solved either by substitution or elimination
- Simultaneous linear and non-linear equations are generally solved by substitution as follows:
  - Step 1: obtain an equation in one unknown & solve it
  - Step 2: substitute the results from step 1 into the linear equation to find the other unknown
- The points of intersection of two graphs are given by the solution of their simultaneous equations

7. Logarithmic & Exponential Functions

- Definition
  - for $a > 0$ and $a \neq 1$

$$y = a^x \Leftrightarrow x = \log_a y$$

- For $\log_a y$ to be defined

$y > 0$ and $a > 0, a \neq 1$

- When the logarithms are defined

$$\log_a 1 = 0 \qquad \log_a b + \log_a c \equiv \log$$

$$\log_a a = 1 \qquad \log_a b - \log_a c \equiv \log$$

$$\log_a b \equiv \frac{\log b}{\log a} \qquad \log_a b^n \equiv n \log$$

- When solving logarithmic equations, check solution with original equation and discard any solutions that causes logarithm to be undefined
- Solution of $a^x = b$ where $a \neq -1,\ 0,\ 1$
- If b can be easily written as a^n, then

$$a^x = a^n \Rightarrow x = n$$

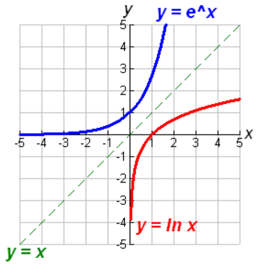
- Otherwise take logarithms on both sides, i.e.

$$\log a^x = \log b \text{ and so } x = \frac{\log b}{\log a}$$

- $\log \Rightarrow \log_{10}$
- $\ln \Rightarrow \log_e$
- *Change of base rule:*

$$\log_a(x) = \frac{\log_b(x)}{\log_a(x)}$$

Logarithmic & Exponential Graphs



8. Straight Line Graphs

- Equation of a straight line:

$$y = mx + c$$

$$y - y_1 = m(x - x_1)$$

- Gradient:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- Length of a line segment:

$$\text{Length} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- Midpoint of a line segment:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

- Point on line segment with ratio m:n

$$\left(\frac{\text{n}x_1 + mx_2}{m + n}, \frac{ny_1 + my_2}{m + n}\right)$$

- Parallelogram:
  - ABCD is a parallelogram $\Longleftrightarrow$ diagonals AC and BD have a common midpoint
  - Special parallelograms = rhombuses, squares, rectangles
- Special gradients:
  - Parallel lines: $m_1 = m_2$
  - Perpendicular lines: $m_1 m_2 = -1$
- **Perpendicular bisector:** line passes through midpoint
- To work out point of intersection of two lines/curves, solve equations simultaneously
- **Find Tangent:** Once the gradient is obtained, substitute the point into the slope-intercept form to get c and the equation.
- **Find normal:** Obtain the gradient by taking the negative reciprocal (see perpendicular gradients). Once the gradient is obtained, substitute the point (original point) into the slope-intercept form to get c and the equation.
- **Find Area,** using two methods
- **Straight Line graphs:** find variables when an equation that does not involve x and y but rather other forms of x and y example: (x^3) or $ln(y)$. This is represented as a straight line.

Mostly in the form $y = ax^n$ or $y = Ab^n$, that must be converted to the form $y = mx + c$.

9. Circular Measure

- Radian measure:

$$\pi = 180^{\circ} \qquad 2\pi = 360^{\circ}$$

Degree to Rad = $\times\frac{\pi}{180}$ Rad to Degree $= \times\frac{180}{\pi}$

- Arc length:

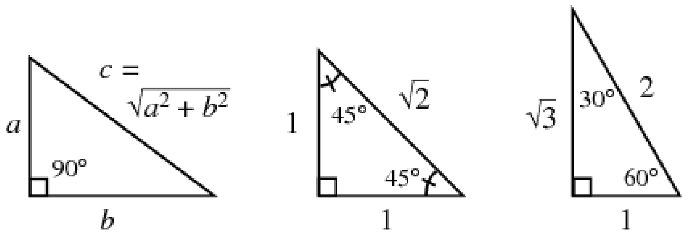
$$s = r\theta$$

- Area of a sector:

$$A = \frac{1}{2}r^2\theta$$

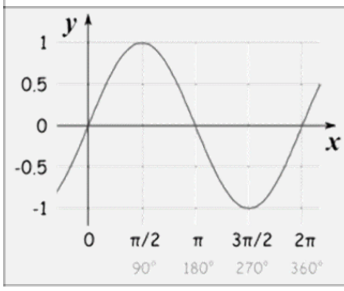
10. Trigonometry

- Trigonometric ratio of special angles:

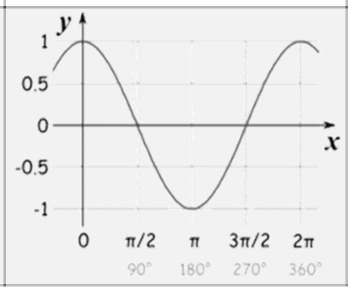


general right triangle *isosceles right triangle* *30–60–90° triangle*

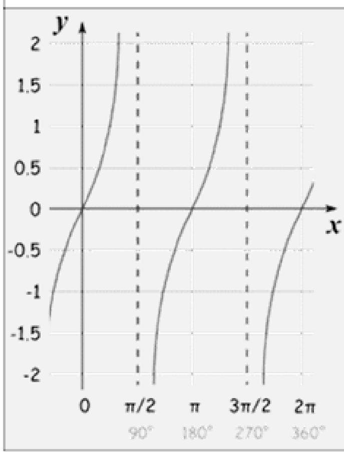
SINE CURVE



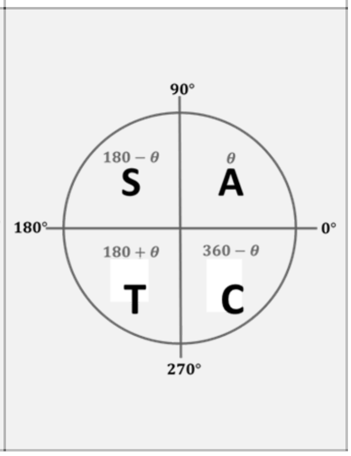
COSINE CURVE



TANGENT CURVE



CAST DIAGRAM



- **Trigonometric ratios:**

$$\sec\theta = \frac{1}{\cos\theta} \qquad \mathrm{cosec}\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

- **Trigonometric identities:**

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \sin^2\theta + \cos^2\theta = 1$$

$$\cot^2\theta + 1 = \mathrm{cosec}^2\theta \qquad \tan^2\theta + 1 = \sec^2\theta$$

- **Sketching trigonometric graphs:**

$$y = a(\ldots bx + c) - d$$

changes amplitude

increases no. of cycles

alters x-axis by $-c$

alters y-axis by d

11. Permutations & Combinations

- Basic counting principle: to find the number of ways of performing several tasks in succession, multiply the number of ways in which each task can be performed: e.g. $5 \times 4 \times 3 \times 2$
- Factorial: $n! = n \times (n-1) \times (n-2) \ldots \times 3 \times 2 \times 1$
  - NOTE: $0! = 1$
- **Permutations:**
  - The number of ordered arrangements of r objects taken from n unlike objects is:

$$^nP_r = \frac{n!}{(n-r)!}$$

- Order **matters**
- **Combinations:**
  - The number of ways of selecting r objects from n unlike objects is:

$$^nC_r = \frac{n!}{r!\,(n-r)!}$$

- Order **does not** matter

12. Series

12.1. Binomial Expansion

- The binomial theorem allows expansion of any expression in the form $(a+b)^n$

$$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \ldots + {}^nC_ny^n$$

- e.g. Expand $(2x-1)^4$

$$(2x-1)^4 = {}^4C_0(2x)^4 + {}^4C_1(2x)^3(-1) + {}^4C_2(2x)^2(-1)^2 + {}^4C_3(2x)(-1)^3 + {}^4C_4(-1)^4$$
$$= 1(2x)^4 + 4(2x)^3(-1) + 6(2x)^2(-1)^2 + 4(2x)(-1)^3 + 1(-1)^4$$
$$= 16x^4 - 32x^3 + 24x^2 - 8x + 1$$

- The powers of x are in descending order

12.2. Sequences & Series

Arithmetic Progression

- A sequence made by adding the same value each time.
- A common difference d is added or subtracted (n-1) times
- General form: $U_n = a + (n-1)d$
- Where n is the number of the term, a (U_1) is the first term and d is the common difference
- Formula for the sum of the first n terms between u_{start} to u_{end}

$$S_n = \frac{n}{2}(u_{start} + u_{end})$$

- **Example:**
  Sequence: 1,2,3,4,5,6
  Sum: 21

Geometric Progression

- A sequence made by multiplying by the same value each time.
- A common ration r is multiplied or divided (n-1) times
- General form: $U_n = ar^{n-1}$
- Where n is the number of the term, a is the first term and r is the common ratio

Example:
Sequence: 2, 4, 8, 16, 32
Sum: 62

- Formula for the sum of the first n numbers of a geometric series

$$S_n = a_1 \times \frac{1-r^n}{1-r}$$

Sum to infinity

- Where the common ratio satisfies the condition: $-1 < r < 1$, it is an infinite geometric progression (convergent progression)

$$S_\infty = a_1 \times \frac{1}{1-r}$$

13. Vectors in 2 Dimensions

- **Position vector:** position of point relative to origin, $\overrightarrow{OP}$
- **Forms of vector:**

$\begin{pmatrix} a \\ b \end{pmatrix}$ $\overrightarrow{AB}$ p $a\mathbf{i} + b\mathbf{j}$

- **Parallel vectors:** same direction but different magnitude
- Generally, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
- Magnitude $= \sqrt{i^2 + j^2}$
- **Unit vectors:** vectors of magnitude 1
  - Examples: consider vector $\overrightarrow{AB}$

$$\overrightarrow{AB} = 2\mathbf{i} + 3\mathbf{j}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{13}$$

$$\therefore Unit\ vector\ = \frac{1}{\sqrt{13}}(2\mathbf{i} + 3\mathbf{j})$$

- **Collinear vectors:** vectors that lie on the same line
- Velocity Vecotr:

$$\begin{pmatrix} a \\ b \end{pmatrix}$$

- **Getting velocity from speed:** Find k to get velocity based on speed

$$k \times \left|\begin{pmatrix} a \\ b \end{pmatrix}\right| = speed$$

- **Point of intersection:**

Object 1 = $\begin{pmatrix} \text{initial x} \\ \text{initial y} \end{pmatrix} + t\begin{pmatrix} a \\ b \end{pmatrix}$

Object 2 = $\begin{pmatrix} \text{initial x} \\ \text{initial y} \end{pmatrix} + t\begin{pmatrix} c \\ d \end{pmatrix}$

Object 1 = Object 2 at time t. If both x and y are not same at intersection time then they will never meet.

14. Differentiation & Integration

14.1. Differentiation

FUNCTION	1ST DERIVATIVE	2ND DERIVATIVE
$y = x^n$	$\frac{dy}{dx} = nx^{n-1}$	$\frac{d^2y}{dx^2} = n(n-1)x^{n-2}$

INCREASING FUNCTION	DECREASING FUNCTION
$\frac{dy}{dx} > 0$	$\frac{dy}{dx} < 0$

- **Stationary point:** equate first derivative to zero

$$\frac{dy}{dx} = 0$$

- **2nd Derivative:** finds nature of the stationary point
  - If $\frac{d^2y}{dx^2} > 0 \rightarrow$ minimum stationary point
  - If $\frac{d^2y}{dx^2} < 0 \rightarrow$ maximum stationary point
- **Chain rule:**

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

- **Product rule:**

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

- **Quotient rule:**

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Special Differentials

$$\frac{dy}{dx}(\sin ax) = a\cos ax$$

$$\frac{dy}{dx}(\cos ax) = -a\sin ax$$

$$\frac{dy}{dx}(\tan ax) = a\sec^2 ax$$

$$\frac{dy}{dx}\left(e^{ax+b}\right) = ae^{ax+b}$$

$$\frac{dy}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{dy}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}$$

- **Related rates of change:**
  - If x and y are related by the equation $y = f(x)$, then the rates of change $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are related by:

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

- **Small changes:**
  - If $y = f(x)$ and small change δx in x causes a small change δy in y, then

$$\delta y \approx \left(\frac{dy}{dx}\right)_{x=k} \times \delta x$$

14.2. Integration

$$\int ax^n = a\frac{x^{n+1}}{(n+1)} + c$$

$$\int (ax+b)^n = \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

- **Definite integral:** substitute coordinates/values & find c
- **Indefinite integral:** has c (constant of integration)

- **Integrating by parts:**

$$\int u\frac{\mathrm{dv}}{\mathrm{dx}}\,\mathrm{dx} = \mathrm{uv} - \int v\frac{\mathrm{du}}{\mathrm{dx}}\mathrm{dx}$$

- What to make u: **LATE**

Logs	Algebra	Trig	e

- **To find area under the graph (curve and x-axis):**
  - Integrate curve
  - Substitute boundaries of x
  - Subtract one from another (ignore c)

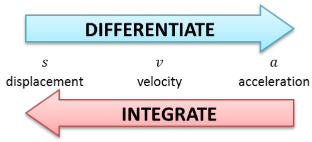
$$\int_a^b \mathrm{y\tilde{d}x}$$

- **To find area between curve and y-axis:**
  - Make x subject of the formula
  - Follow above method using y-values instead of x-values

Special Integrals

- $\int \sin(\mathrm{ax}+b) = -\frac{1}{a}\cos(\mathrm{ax}+b)+c$
- $\int \cos(\mathrm{ax}+b) = \frac{1}{a}\sin(\mathrm{ax}+b)+c$
- $\int \sec^2(\mathrm{ax}+b) = \frac{1}{a}\tan(\mathrm{ax}+b)+c$
- $\int \frac{1}{\mathrm{ax}+b} = \frac{1}{a}\ln|\mathrm{ax}+b|+c$
- $\int e^{\mathrm{ax}+b} = \frac{1}{a}e^{\mathrm{ax}+b}+c$

14.3. Kinematics



- Particle at instantaneous rest, $v=0$
- Maximum displacement from origin, $v=0$
- Maximum velocity, $a=0$

CAIE IGCSE
Add Maths (0606)