CAMBRIDGE INTERNATIONAL EXAMINATIONS

Pre-U Certificate

MARK SCHEME for the May/June 2013 series

1348 FURTHER MATHEMATICS

1348/01 Paper 1 (Further Pure Mathematics), maximum raw mark 120

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

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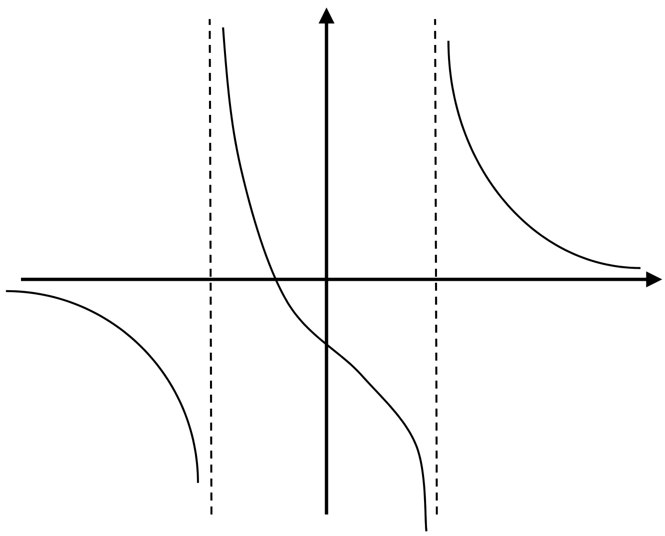
1		$x^2 - 6x + 12 \equiv (x-3)^2 + 3$	B1
		$\displaystyle\int_2^6 \frac{1}{\left(\sqrt{3}\right)^2 + (x-3)^2}\,\mathrm{d}x = \left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x-3}{\sqrt{3}}\right)\right]_2^6$ **M1** $\tan^{-1}$ 1st **A1** $\left(\frac{x-3}{\sqrt{3}}\right)$	M1 A1
		$= \dfrac{1}{\sqrt{3}}\left(\dfrac{\pi}{3} - \left(-\dfrac{\pi}{6}\right)\right) = \dfrac{\pi}{2\sqrt{3}}$	A1
			[4]
2		$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + \ldots$ from the Formula Book	
		$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5 - \ldots$	B1
		$\frac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right) = \frac{1}{2}\{\ln(1+x) - \ln(1-x)\}$	M1
		$= \frac{1}{2} \times 2\left\{x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \ldots\right\} = \tanh^{-1}x$ from the Formula Book	A1
		$\ln(1+x)$ valid for $-1 < x \le 1$ and so $\ln(1-x)$ is valid for $-1 \le x < 1$ so LHS valid for $-1 < x < 1$, which matches the range for RHS	B1
			[4]
3	(i)	$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\left(x^2-4\right).1 - (x+1).2x}{\left(x^2-4\right)^2}$ Use of quotient rule; correct unsimplified	M1 A1
		$= -\dfrac{\left(x^2+2x+4\right)}{\left(x^2-4\right)^2} = -\dfrac{(x+1)^2+3}{\left(x^2-4\right)^2}$ or clear explanation this is < 0	E1
		ALT: $y = \dfrac{3/4}{x-2} + \dfrac{1/4}{x+2} \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{-3/4}{(x-2)^2} + \dfrac{-1/4}{(x+2)^2} < 0$	
			[3]

3	(ii)	Asymptotes $y = 0$ Stated or clear from graph $x = \pm 2$ Stated or clear from graph	B1 B1
		Crossing-points $(0, -\frac{1}{4})$ and $(-1, 0)$ Noted or clearly shown on graph	B1 B1
		 3 regions All correct (incl. no TPs)	 M1 A1 [6]
4	(i)	$\mathbf{d_1} \times \mathbf{d_2}$ attempted $= 14\mathbf{i} + 35\mathbf{j} - 21\mathbf{k}$ (**ALT:** Use of 2 scalar prods. & attempt to get 2 components in terms of the 3rd)	M1 A1 [2]
	(ii)	Sh. Dist. $= \lvert (\mathbf{b} - \mathbf{a}).\ \hat{\mathbf{n}} \rvert$ $(\mathbf{b} - \mathbf{a}) = \pm(\mathbf{i} + 3\mathbf{j} - 7\mathbf{k})$ $\hat{\mathbf{n}} = \frac{1}{\sqrt{38}}(2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k})$ **ft**	M1 B1 B1
		$= \frac{1}{\sqrt{38}}(2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}) \bullet (\mathbf{i} + 3\mathbf{j} - 7\mathbf{k}) = \frac{1}{\sqrt{38}}\lvert 2 + 15 + 21 \rvert$ **ft** scalar prod.	B1
		$= \sqrt{38}$ **cao**	A1
		ALT: Solving $\begin{pmatrix}2\\4\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\4\end{pmatrix} - \begin{pmatrix}1\\1\\10\end{pmatrix} - \mu\begin{pmatrix}9\\-3\\1\end{pmatrix} = k\begin{pmatrix}2\\5\\-3\end{pmatrix}$ to find closest points on line, (3, 6, 7) from $\lambda = 1$ and (1, 1, 10) from $\mu = 0$ giving $k = 1$ and Sh.D. $= \sqrt{38}$	 [5]

5	**(i)**	$z^n - \frac{1}{z^n} = (\cos n\theta + \text{i}.\sin n\theta) - (\cos[-n\theta] - \text{i}.\sin[-n\theta])$ De Moivre's Thm. used for at least z^n	**M1**
		$= \cos n\theta + \text{i}.\sin n\theta - (\cos n\theta - \text{i}.\sin n\theta) = 2\text{i} \sin n\theta$ Given answer obtained from 2 correct uses of de Moivre's Thm. and correct trig.	**A1** **[2]**
	(ii)	$\left(z - \frac{1}{z}\right)^5 = 32\text{i} \sin^5\theta$	**M1**
		$= z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$ Use of binomial expansion	**M1**
		$= \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$ Pairing up terms	**M1**
		$= 2\text{i} \sin 5\theta - 10\text{i} \sin 3\theta + 20\text{i} \sin\theta$ Use of (i)'s result (×3)	**M1**
		$\Rightarrow \sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$	**A1** **[5]**
6	**(i)**	$r = 1 + r\sin\theta \Rightarrow \sqrt{x^2 + y^2} = 1 + y$	**M1 M1**
		Squaring and cancelling: $x^2 + y^2 = y^2 + 2y + 1 \Rightarrow y = \frac{1}{2}(x^2 - 1)$	**A1** **[3]**
	(ii)	Parabola All correct: Crossing-points at (± 1, 0) and $(0, -\frac{1}{2})$	**M1 A1** **[2]**
	(iii)	$\int_{\pi}^{2\pi} \frac{1}{(1 - \sin\theta)^2}\,\text{d}\theta = 2 \times \int_{\pi}^{2\pi} \frac{1}{2}r^2\,\text{d}\theta$ Recognition that this is related to area	**M1**
		$= -2\int_{-1}^{1} \frac{1}{2}(x^2 - 1)\,\text{d}x$ Matching up with parabola-related region	**M1**
		$= -\left[\frac{x^3}{3} - x\right]_{-1}^{1} = \frac{4}{3}$ Ignore –ve answer	**A1** **[3]**

7	**(i)**	$x^3 + y^3 = (x+y)^3 - 3xy(x+y)$ or equivalent	**M1 A1** **[2]**
	(ii)	**(a)** $\alpha + \beta$ $(= 3)$ and $\alpha\beta$ $(= \frac{8}{9})$ substd. into (i)'s result **ft** $\Rightarrow$ $\alpha^3 + \beta^3 = 19$	**M1 A1** **[2]**
		(b) $9t^2 - 27t + 8 = 0 \Rightarrow (3t-1)(3t-8) = 0 \Rightarrow \alpha, \beta = \frac{1}{3}, \frac{8}{3}$ Then $\alpha^3 + \beta^3 = 19 = \left(\frac{1}{3}\right)^3 + \left(\frac{8}{3}\right)^3$ Explicit statement required	**M1 A1** **A1** **[3]**
8	**(i)**	**(a)** $x \in G \Rightarrow \exists\, x^{-1} \in G$ and pre-multiplying by this (or x in the $\Leftarrow$ case) gives the result (NB **Both** directions must be dealt with)	**B1** **B1** **[2]**
		(b) Since each xg_i is distinct, and there are n of them, the set xG is just a permutation of the elements of G **OR** mention that it is just a row of the group table and hence contains a permutation of the elements of G	**B1** **[1]**
	(ii)	Multiply all elements together: $xg_1\, xg_2\, xg_3 \ldots xg_n = g_1\, g_2\, g_3 \ldots\ g_n$ (Since G is abelian) $\Rightarrow$ $x^n.(g_1\, g_2\, g_3 \ldots\ g_n) = (g_1\, g_2\, g_3 \ldots\ g_n)$ Since $g_1\, g_2\, g_3 \ldots\ g_n$ is an element of G, it has an inverse; Pre/post-mult^g. by this inverse then gives $x^n = e$	**E1** **E1** **E1** **[3]**
	(iii)	**(a)** Elements may have an order which divides into (is a factor of) n	**B1** **[1]**
		(b) Because the change of the order of multns. in $g.g_1\, g.g_2\, g.g_3 \ldots g.g_n = g^n.(g_1\, g_2\, g_3 \ldots\ g_n)$ is only valid in an abelian group	**B1** **[1]**

9		Reflection in $y = x \tan\frac{1}{8}\pi$: $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$ Allow $\cos\left(\frac{1}{4}\pi\right)$'s , etc.	**B1**
		Shear // y-axis, mapping (1, 0) to (1, 2): $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$	**B1**
		Rotation through $\frac{1}{4}\pi$ clockwise about O: $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$	**B1**
		Shear // x-axis, mapping (0, 1) to (–2, 1): $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$	**B1**
		Multiplying them together in this order (from right-to-left) = $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	**M1** **A1**
		Reflection in $y = x$	**M1 A1**
			[8]
		NB 1 Multiplying the matrices in the reverse order scores max. 4 × **B1** + **M0** ; then **B1** for correct $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and **M1** for "Reflection" and **A1** for "in x-axis" **NB 2** Incorrect final matrices automatically lose the last 2 marks	
10	(a)	$y = k\,x\cos x \Rightarrow \frac{dy}{dx} = -k\,x\sin x + k\cos x$ and $\frac{d^2y}{dx^2} = -k\,x\cos x - 2k\sin x$ Attempt at 1st and 2nd derivatives using the *Product Rule*	**M1**
		Substituting both of these into the given DE	**M1**
		$-k\,x\cos x - 2k\sin x + k\,x\cos x = 4\sin x$	
		Comparing terms to evaluate k: $k = -2$	**M1 A1**
		Aux. Eqn. $m^2 + 1 = 0$ solved $\Rightarrow m = \pm\,\mathrm{i}$	**M1 A1**
		Comp. Fn. is $y_C = A\cos x + B\sin x$ **ft** Accept $y_C = Ae^{ix} + Be^{-ix}$ here	**B1**
		G. S. is $y = A\cos x + B\sin x - 2x\cos x$ **ft** provided y_P has no arb. consts. & y_C has 2	**B1**
		Do not accept final answer involving complex numbers	
			[8]

10	(b)(i)	$x=1,\ y=2\ \&\ \frac{dy}{dx}=1 \Rightarrow \left.\frac{d^2y}{dx^2}\right]_{x=1} = -20$	B1
		Differentiating $\frac{d^2y}{dx^2}+y^2\frac{dy}{dx}+xy=5x-19$	M1
		Use of *Product Rule* and implicit differentiation (at least once)	M1
		$\Rightarrow \frac{d^3y}{dx^3}+\left\{y^2\frac{d^2y}{dx^2}+2y\left(\frac{dy}{dx}\right)^2\right\}+\left\{x\frac{dy}{dx}+y\right\}=5 \Rightarrow \left.\frac{d^2y}{dx^2}\right]_{x=1}=78$	A1 A1 A1
		FT "78" from "–20" and also from $\frac{dy}{dx}$ instead of $\left(\frac{dy}{dx}\right)^2$ (both = 1)	**[6]**
	(ii)	Use of $y=y(1)+(x-1).y'(1)+\frac{1}{2}(x-1)^2.y''(1)+\frac{1}{6}(x-1)^3.y'''(1)+\ldots$	M1
		$=2+(x-1)-10(x-1)^2+13(x-1)^3+\ldots$ **ft**	A1
		Substituting $x=1.1$ into this series $\Rightarrow y(1.1)\approx 2.013$ **ft**	M1 A1
			[4]
11	(i)	$(p+iq)^2=(p^2-q^2)+i.2pq$	B1
		Comparing real and imaginary parts: $p^2-q^2=63$ and $2pq=-16$	M1
		Solving simultaneously : $p=\pm 8,\ q=\mp 1$ i.e. $\pm(8-i)^2=63-16i$	M1 A1
			[4]
	(ii)	**(a)** Use of $z^3-(\alpha+\beta+\gamma)z^2+(\alpha\beta+\beta\gamma+\gamma\alpha)z-(\alpha\beta\gamma)=0$	M1
		$A=4-4i,\ B=21-16i,\ C=84$ i.e. $f(z)=z^3-(4-4i)z^2+(21-16i)z-84=0$	A1 A1 A1
			[4]
		(b) Differentiating to get $f'(z)=3z^2-8(1-i)z+(21-16i)$ **OR** $3z^2-2Az+B=0$ **ft**	B1
		Solving $z=\frac{8-8i\pm\sqrt{64(1-2i-1)-12(21-16i)}}{6}$ using the quadratic formula	M1
		$z=\frac{1}{3}\left(4-4i\pm\sqrt{16i-63}\right)=\frac{1}{3}\left(4-4i\pm i\sqrt{63-16i}\right)$	A1
		Use of (i)'s result (on the right thing): $z=\frac{1}{3}\left(4-4i\pm i(8-i)\right)=\frac{5}{3}+\frac{4}{3}i$ or $1-4i$	M1 A1
			[5]

12	(i)	$y'(x) = (2x+1)\,e^{2x}$, $\quad y''(x) = (4x+4)\,e^{2x}$,	**B1 B1**
		$y'''(x) = (8x+12)\,e^{2x}$, $\quad y^{(4)}(x) = (16x+32)\,e^{2x}$	**B1 B1**
			[4]
	(ii)	Conjecture $\dfrac{d^n y}{dx^n} = (2^n x + n.2^{n-1})\,e^{2x}$ One mark each: coefft. of x, constant	**B1 B1**
			[2]
	(iii)	Diffferentiating their conjectured expression (must be linear × e^{2x})	**M1**
		$\dfrac{d^{n+1}y}{dx^{n+1}} = 2 \times (2^n x + n.2^{n-1})\,e^{2x} + 2^n \times e^{2x}$ **FT** max 1/2	**A1 A1**
		$= (2^{n+1}x + (n+1).2^{(n+1)-1})\,e^{2x}$ Shown of correct form	**A1**
		Usual induction round-up/explanation of proof, including clear demonstration that $(n+1)^{\text{th}}$ formula is in the right form.	**E1**
			[5]
13	(i)	**(a)** $1 - \operatorname{sech}^2\theta \equiv \dfrac{\left(e^{\theta}+e^{-\theta}\right)^2 - 4}{\left(e^{\theta}+e^{-\theta}\right)^2} \equiv \dfrac{\left(e^{\theta}-e^{-\theta}\right)^2}{\left(e^{\theta}+e^{-\theta}\right)^2} \equiv \tanh^2\theta$ shown legitimately	**M1** **A1**
		(b) $\dfrac{d}{d\theta}(\tanh\theta) = \dfrac{\left(e^{\theta}+e^{-\theta}\right)\left(e^{\theta}+e^{-\theta}\right) - \left(e^{\theta}-e^{-\theta}\right)\left(e^{\theta}-e^{-\theta}\right)}{\left(e^{\theta}+e^{-\theta}\right)^2} \equiv \operatorname{sech}^2\theta$ from (a)	**M1** **A1**
			[4]
	(ii)	**(a)** $I_n = \int_0^{\alpha} \tanh^{2n-2}\theta.\tanh^2\theta\,d\theta = \int_0^{\alpha} \tanh^{2n-2}\theta\left(1-\operatorname{sech}^2\theta\right)d\theta$	**M1 M1**
		$= I_{n-1} - \left[\dfrac{\tanh^{2n-1}\theta}{2n-1}\right]_0^{\alpha} \Rightarrow I_{n-1} - I_n = \dfrac{(\tanh\alpha)^{2n-1}}{2n-1}$	**M1 A1**
		ALT: $I_{n-1} - I_n = \int_0^{\alpha} \tanh^{2n-2}\theta\left(1-\tanh^2\theta\right)d\theta = \int_0^{\alpha} \tanh^{2n-2}\theta.\operatorname{sech}^2\theta\,d\theta$	**M1 M1**
		$= \left[\dfrac{\tanh^{2n-1}\theta}{2n-1}\right]_0^{\alpha} = \dfrac{(\tanh\alpha)^{2n-1}}{2n-1}$	**M1 A1**
			[4]

13	(ii)	**(b)** $I_0 = \int_0^{\alpha} 1\,d\theta = \alpha = \frac{1}{2}\ln 3$	B1
			[1]
		(c) $(I_{n-1} - I_n) + (I_{n-2} - I_{n-1}) + (I_{n-3} - I_{n-2}) + \ldots + (I_2 - I_3) + (I_1 - I_2) + (I_0 - I_1)$ Use of the *method of differences*	M1
		$= \sum_{r=1}^{n} \frac{(\tanh\alpha)^{2r-1}}{2r-1} = \sum_{r=1}^{n} \frac{\left(\frac{1}{2}\right)^{2r-1}}{2r-1}$ when $\alpha = \frac{1}{2}\ln 3$	A1
		$\Rightarrow I_0 - I_n = \sum_{r=1}^{n} \frac{\left(\frac{1}{2}\right)^{2r-1}}{2r-1}$ Cancellation of terms in the summation	M1
		$\Rightarrow I_n = \frac{1}{2}\ln 3 - \sum_{r=1}^{n} \frac{\left(\frac{1}{2}\right)^{2r-1}}{2r-1}$ **AG**	A1
		Ignoring "method of differences", but opting for a direct iterative approach scores max 3/4 … **M0 M1 A1 A1**	
		As $n \to \infty$, $I_n \to 0$ since $\lvert \tanh \rvert < 1$	E1
		$\Rightarrow \frac{1}{2} ln 3 = \sum_{r=1}^{n} \frac{\left(\frac{1}{2}\right)^{2r-1}}{2r-1} = \frac{\frac{1}{2}}{1} + \frac{\left(\frac{1}{2}\right)^3}{3} + \frac{\left(\frac{1}{2}\right)^5}{5} + \frac{\left(\frac{1}{2}\right)^7}{7} + \ldots$	M1
		$\Rightarrow \ln 3 = 1 + \frac{1}{3.4} + \frac{1}{5.4^2} + \frac{1}{7.4^3} + \ldots = \sum_{r=0}^{\infty} \frac{1}{(2r+1)4^r}$	A1
		Ignoring "method of differences", but opting for a direct iterative approach scores max 3/4 … **M0 M1 A1 A1**	[7]