Cambridge Assessment International Education
Cambridge Pre-U Certificate

FURTHER MATHEMATICS (SHORT COURSE) **1348/01**

Paper 1 Further Pure Mathematics **May/June 2018**

MARK SCHEME

Maximum Mark: 120

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

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Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1:

Marks must be awarded in line with:

- the specific content of the mark scheme or the generic level descriptors for the question
- the specific skills defined in the mark scheme or in the generic level descriptors for the question
- the standard of response required by a candidate as exemplified by the standardisation scripts.

GENERIC MARKING PRINCIPLE 2:

Marks awarded are always **whole marks** (not half marks, or other fractions).

GENERIC MARKING PRINCIPLE 3:

Marks must be awarded **positively**:

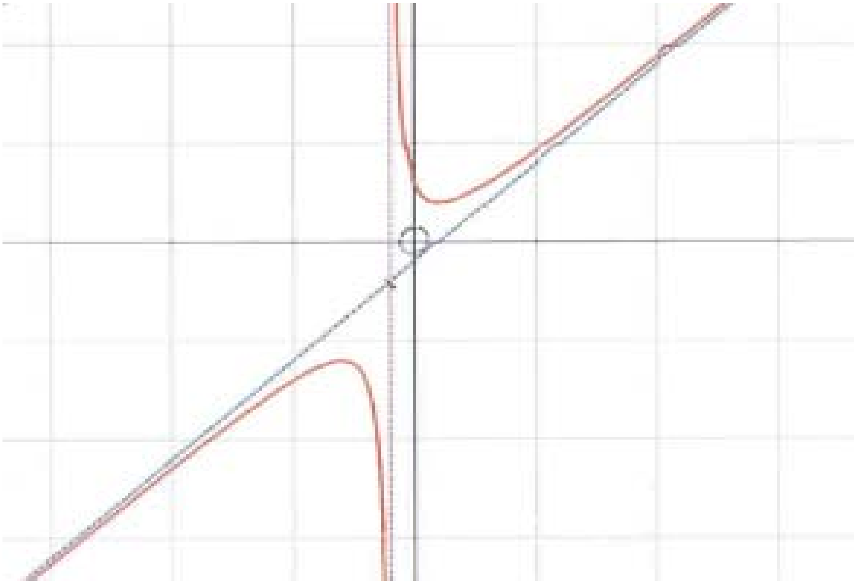
- marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate
- marks are awarded when candidates clearly demonstrate what they know and can do
- marks are not deducted for errors
- marks are not deducted for omissions
- answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.

GENERIC MARKING PRINCIPLE 4:

Rules must be applied consistently e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.

GENERIC MARKING PRINCIPLE 5: Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).
GENERIC MARKING PRINCIPLE 6: Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Question	Answer	Marks	Guidance
1(i)	$\frac{3}{(3r-1)(3r+2)} \equiv \frac{1}{3r-1} - \frac{1}{3r+2}$	M1	**M1** for attempt at PFs
		A1	
1(ii)	$\sum_{r=1}^{n} \frac{3}{(3r-1)(3r+2)} = \sum_{r=1}^{n} \frac{1}{3r-1} - \sum_{r=1}^{n} \frac{1}{3r+2}$	M1	**M1** for splitting into the difference of two series or a series of paired differences
	$= \left(\frac{1}{2} + \frac{1}{5} + \frac{1}{8} + \ldots + \frac{1}{3n-1}\right) - \left(\frac{1}{5} + \frac{1}{8} + \ldots + \frac{1}{3n-1} + \frac{1}{3n+2}\right)$		
	$= \frac{1}{2} - \frac{1}{3n+2}$	A1	**Given Answer** must come from fully correct working fully shown
1(iii)	As $n \to \infty$, $\frac{1}{3n+2} \to 0$ so $S_\infty = \frac{1}{6}$	B1	**CAO** (Limiting argument not required)
2(i)	VA $x = -1$	B1	
	$y = \frac{x(x+1) - (x+1) + 4}{x+1} = x - 1 + \frac{4}{x+1}$	M1	For attempt at long-division (or equivalent)
	so OA is $y = x - 1$	A1	Ignore errors with the remainder term Condone $y \to x - 1$ but not $y \neq x - 1$ Withhold this A1 if any extra asymptotes (e.g. a HA) given
	$\frac{dy}{dx} = \frac{(x+1).2x - (x^2+3).1}{(x+1)^2} = \frac{x^2+2x-3}{(x+1)^2}$	M1	For differentiating. **ALT** $\frac{dy}{dx} = 1 - \frac{4}{(x+1)^2}$
	Setting $\frac{dy}{dx} = 0$ and solving $\Rightarrow$ $x = 1$ or -3	M1 A1	
	$y = 2$ or -6	A1	Give one A1 for a correct (x, y) pair

Question	Answer	Marks	Guidance
2(ii)	Two asymptotes (one VA and one OA)	**B1**	**FT** (The OA must actually be an asymptote)
	Two branches in correct "quadrants"	**B1**	
	TPs in approx. correct places and (0, 3) noted somewhere	**B1**	**FT** sensible co-ords of TPs relative to curve
			
3(i)	$\left\lvert \dfrac{z_1}{z_2} \right\rvert = \sqrt{2}$	**B1**	Modulus
	$\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) = \dfrac{17\pi}{24}$	**M1 A1**	Argument

Question	Answer	Marks	Guidance
3(ii)	$\arg(z_3) = \dfrac{17n\pi}{24}$	**B1**	**FT**
	Require $\dfrac{17n\pi}{24}$ to be an even multiple of $\pi \Rightarrow n_{\min} = 48$	**M1 A1**	**FT** unless trivial
	so that $z_3 = 2^{24}$ or 16 777 216	**A1**	**CAO**
4(i)	$r = \dfrac{3}{10}e^{\frac{3}{4}\theta} \Rightarrow \dfrac{dr}{d\theta} = \dfrac{9}{40}e^{\frac{3}{4}\theta}$	**M1**	Derivative of r found and attempt at $r^2 + \left(\dfrac{dr}{d\theta}\right)^2$
	$r^2 + \left(\dfrac{dr}{d\theta}\right)^2 = \dfrac{144}{1600}e^{\frac{3}{2}\theta} + \dfrac{81}{1600}e^{\frac{3}{2}\theta} = \dfrac{9}{64}e^{\frac{3}{2}\theta}$	**A1**	Accept any equivalent fractions
	$L(\alpha) = \int \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}\, d\theta = \int \dfrac{3}{8}e^{\frac{3}{4}\theta}\, d\theta$	**M1**	Attempted use of appropriate arc-length formula (ignore limits for now)
	$= \left[\dfrac{1}{2}e^{\frac{3}{4}\theta}\right]$	**A1**	Correct integration of $ae^{k\theta}$ term
	$= \dfrac{1}{2}\left(e^{\frac{3}{4}\alpha} - 1\right)$	**A1**	**Given Answer** correctly established

Question	Answer	Marks	Guidance
4(ii)	$\frac{3}{10}e^{\frac{3}{4}\beta} = \frac{1}{2}e^{\frac{3}{4}\beta} - \frac{1}{2} \Rightarrow \frac{1}{5}e^{\frac{3}{4}\beta} = \frac{1}{2}$	**M1**	Solving this equation
	$\Rightarrow \beta = \frac{4}{3}\ln\left(\frac{5}{2}\right)$	**A1**	Allow 1.22(172…)
5	$\exp\left\{\int \tanh x \, dx\right\} = \exp\left\{\int \frac{\sinh x}{\cosh x} dx\right\} = \exp\{\ln(\cosh x)\} = \cosh x$	**M1 A1**	Attempt at Integrating Factor; correct
	DE becomes $\cosh x \frac{dy}{dx} + y \sinh x = 2\cosh^2 x$	**B1**	**FT**
	$\Rightarrow y \cosh x = \int (1 + \cosh 2x) \, dx = x + \frac{1}{2}\sinh 2x \ (+ C)$	**B1** **M1 A1**	Integrating both sides: LHS RHS
	Use of $x = \ln 2, y = \frac{3}{4}$ to evaluate C $(= -\ln 2)$	**M1**	
	$y = \frac{x - \ln 2}{\cosh x} + \sinh x$	**A1**	In any correct $y = \ldots$ form
6(i)	$1 + R_1 = \frac{r_1 + r_2 + r_3}{r_1} = \frac{3}{r_1}$ since $\sum r_i = \frac{12}{4} = 3$	**B1**	
	Also, $1 + R_2 = \frac{3}{r_2}$ and $1 + R_3 = \frac{3}{r_3}$	**B1**	

Question	Answer	Marks	Guidance
6(ii)	$1+y=\frac{3}{x}$ the required substitution	**B1**	Any arrangement
	$x=\frac{3}{y+1}$ substituted into $4x^3-12x^2+9x-16=0$	**M1**	
	$\frac{4\times 27}{(y+1)^3}-\frac{12\times 9}{(y+1)^2}+\frac{9\times 3}{(y+1)}-16=0$	**A1**	Correct unsimplified
	$\Rightarrow\ 108-108(y+1)+27(y+1)^2-16(y+1)^3=0$	**M1**	Multiplying by $(y+1)^3$
	$\Rightarrow\ 108-108y-108+27y^2+54y+27\ -16y^3-48y^2-48y-16=0$	**M1**	Expanding brackets and collecting up terms
	$\Rightarrow\ 16y^3+21y^2+102y-11=0$	**A1**	Must have integer coefficients (multiples accepted)
	ALT. I $\sum R_i=\frac{\sum r_i^2 r_j}{r_1r_2r_3}=\frac{\left(\sum r_i\right)\left(\sum r_ir_j\right)-3r_1r_2r_3}{r_1r_2r_3}=\frac{(3)\left(\frac{9}{4}\right)-3\times 4}{4}=\frac{21}{16}$ **M1** (complete attempt) **A1**		
	$\sum R_iR_j=\frac{\sum r_i^2 r_j+\sum r_i^3+3r_1r_2r_3}{r_1r_2r_3}=\frac{-\frac{21}{4}+\frac{75}{4}+3\times 4}{4}=\frac{102}{16}$ **M1** (complete attempt) **A1**		
	$\prod R_i=\frac{\sum r_i^2 r_j+2r_1r_2r_3}{r_1r_2r_3}=\frac{-\frac{21}{4}+8}{4}=\frac{11}{16}$ **M1** (complete attempt) **A1** all coeffts. correct and final statement of equation; must have integer coefficients (multiples accepted)		

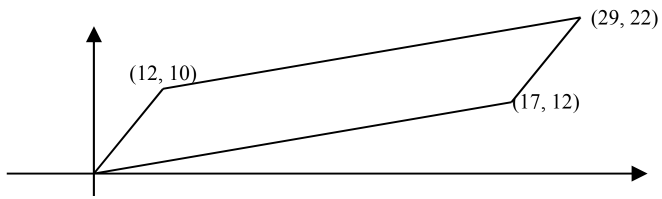
Question	Answer	Marks	Guidance
	ALT. II From calculator, $r_1 = 2.714\,014\,591$, $r_{2,3} = 0.142\,992\,704\,4 \pm 1.205\,563\,982\,\mathrm{i}$	**B1**	
	Using $R = \frac{3}{r} - 1$, $R_1 = 0.105\,373\,570\,8$, $R_{2,3} = -0.708\,936\,785\,4 \mp 2.453\,938\,678\,\mathrm{i}$	**M1**	
	Then $R_1 + R_2 + R_3 = -1.3125 = -\frac{21}{16}$, $R_1R_2 + R_2R_3 + R_3R_1 = 6.375 = \frac{102}{16}$, $R_1R_2R_3 = \frac{11}{16}$ **NB** Calc. gives $6.673\,812\,802 \approx \frac{107}{16}$	**M1 A1 A1 A1**	
	$\Rightarrow 16y^3 + 21y^2 + 102y - 11 = 0$ Final A1 can only be given for correct final statement of eqn. with integer coeffts. (multiples accepted)		
7(i)(a)	$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right]_1 = 0$	**B1**	
7(i)(b)	Diffg. $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x^2y = x$ implicitly	**M1**	Including correct use of the *Product Rule*
	$\Rightarrow \frac{\mathrm{d}^3y}{\mathrm{d}x^3} + \left(x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy\right) = 1 \Rightarrow \left.\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right]_1 = -2$	**A1**	
7(ii)	$y(x) = y(1) + \frac{y'(1)}{1!}(x-1) + \frac{y''(1)}{2!}(x-1)^2 + \frac{y'''(1)}{3!}(x-1)^3 + \ldots$	**M1**	Attempt at *Taylor Series*, correct in principle
	$= 1 + (x-1) - \frac{1}{3}(x-1)^3 \ldots$	**A1**	**FT** provided cubic term is non-zero
	$y(1.1) = 1.0997$ to 4 d.p.	**B1**	**CSO** (actual value is 1.0996 to 4 d.p.)

Question	Answer	Marks	Guidance
8(i)	$a = 6,\ b = 4$	**B1**	
8(ii)	For $n = 1$, LHS = $1^5 = 1$ and RHS = $\frac{1}{6}.2^3 - \frac{1}{12}.2^2 = \frac{4}{3} - \frac{1}{3} = 1$ so that the result is true for $n = 1$	**B1**	Both sides must be established
	Assume that $\sum_{r=1}^{k} r^5 = \frac{1}{6}k^3(k+1)^3 - \frac{1}{12}k^2(k+1)^2$	**M1**	Induction hypothesis clearly stated somewhere
	Then $\sum_{r=1}^{k+1} r^5 = \frac{1}{6}k^3(k+1)^3 - \frac{1}{12}k^2(k+1)^2 + (k+1)^5$	**M1**	Attempt at S_{k+1} with S_k used
	$= \frac{1}{6}k^3(k+1)^3 - \frac{1}{12}k^2(k+1)^2$ $+\frac{1}{6}(k+1)^3\left[6(k+1)^2+2\right] - \frac{1}{12}(k+1)^2\left[4(k+1)\right]$	**M1**	Use of (i)'s result with $m = k + 1$ for the $(k + 1)^5$ term
	$= \frac{1}{6}(k+1)^3\left[k^3+6\left(k^2+2k+1\right)+2\right] - \frac{1}{12}(k+1)^2\left[k^2+4(k+1)\right]$	**M1**	Terms collected appropriately
	$= \frac{1}{6}(k+1)^3(k+2)^3 - \frac{1}{12}(k+1)^2(k+2)^2$	**A1**	Legitimately shown so
	Hence result true for $n = k \Rightarrow$ result true for $n = k + 1$. Since result true for $n = 1$, it follows that it is true for $n = 2$, $n = 3$, etc. and the result is true for all positive integers n by induction	**E1**	Induction process clearly explained: minimum requirement is (P_1 ✓) and (P_k ✓ $\Rightarrow P_{k+1}$ ✓)

Question	Answer	Marks	Guidance
8(ii)	**Alt. I** For $n = 1$, LHS = $1^5 = 1$ and RHS = $\frac{1}{6}.2^3 - \frac{1}{12}.2^2 = \frac{4}{3} - \frac{1}{3} = 1$ so that the result is true for $n = 1$	**B1**	Both sides must be established
	Assume that $\sum_{r=1}^{k} r^5 = \frac{1}{6}k^3(k+1)^3 - \frac{1}{12}k^2(k+1)^2$	**M1**	Induction hypothesis clearly stated somewhere
	Then $\sum_{r=1}^{k+1} r^5 = \frac{1}{6}k^3(k+1)^3 - \frac{1}{12}k^2(k+1)^2 + (k+1)^5$	**M1**	Attempt at S_{k+1} with S_k used
	$= \frac{1}{12}(k+1)^2(2k^4 + 14k^3 + 35k^2 + 36k + 12)$	**M1**	Factorising out the $(k+1)^2$
	$= \frac{1}{12}(k+1)^2(k^2 + 4k + 4)(2k^2 + 6k + 3)$		
	$= \frac{1}{12}(k+1)^2(k+2)^2(2(k+1)(k+2) - 1)$	**M1**	Factorising and splitting the final factor suitably
	$= \frac{1}{6}(k+1)^3(k+2)^3 - \frac{1}{12}(k+1)^2(k+2)^2$	**A1**	Legitimately shown so
	Hence result true for $n = k \Rightarrow$ result true for $n = k + 1$. Since result true for $n = 1$, it follows that it is true for $n = 2$, $n = 3$, etc. and the result is true for all positive integers n by induction	**E1**	Induction process clearly explained: minimum requirement is (P_1 ✓) and (P_k ✓ $\Rightarrow P_{k+1}$ ✓)

Question	Answer	Marks	Guidance
8(ii)	**Alt. II** For $n = 1$, LHS $= 1^5 = 1$ and RHS $= \frac{1}{6}.2^3 - \frac{1}{12}.2^2 = \frac{4}{3} - \frac{1}{3} = 1$ so that the result is true for $n = 1$	**B1**	Both sides must be established
	Assume that $\sum_{r=1}^{k} r^5 = \frac{1}{6}k^3(k+1)^3 - \frac{1}{12}k^2(k+1)^2$	**M1**	Induction hypothesis clearly stated somewhere
	Then $\sum_{r=1}^{k+1} r^5 = \frac{1}{6}k^3(k+1)^3 - \frac{1}{12}k^2(k+1)^2 + (k+1)^5$	**M1**	Attempt at S_{k+1} with S_k used
	$= \frac{1}{6}k^6 + \frac{3}{2}k^5 + \frac{65}{12}k^4 + 10k^3 + \frac{119}{12}k^2 + 5k + 1$	**M1**	Multiplying it all out and collecting up terms
	RHS $= \sum_{r=1}^{k+1} r^5 = \frac{1}{6}(k+1)^3(k+2)^3 - \frac{1}{12}(k+1)^2(k+2)^2$	**M1**	Full attempt to multiply out the expected S_{k+1}
	$= \frac{1}{6}k^6 + \frac{3}{2}k^5 + \frac{65}{12}k^4 + 10k^3 + \frac{119}{12}k^2 + 5k + 1$	**A1**	Convincingly shown so, both sides
	Hence result true for $n = k \Rightarrow$ result true for $n = k + 1$. Since result true for $n = 1$, it follows that it is true for $n = 2$, $n = 3$, etc. and the result is true for all positive integers n by induction	**E1**	Induction process clearly explained: minimum requirement is (P_1 ✓) and (P_k ✓ $\Rightarrow P_{k+1}$ ✓)
9(i)	$\cos 3\theta = \text{Re}(\cos 3\theta + \text{i} \sin 3\theta) = \text{Re}(c + \text{i}\, s)^3$	**M1**	Use of *De Moivre's Theorem* (with $n = 3$) at some stage
	$= \text{Re}\left(c^3 + 3c^2.\text{i}s + 3c.\text{i}^2s^2 + \text{i}^3s^3\right)$	**M1**	Binomial expansion (only real terms need be seen)
	$= c^3 - 3c(1 - c^2) = 4c^3 - 3c$	**A1**	**Given Answer** legitimately obtained, fully supported

Question	Answer	Marks	Guidance
9(ii)	$\cos 3\theta = \frac{1}{2}\sqrt{3} \Rightarrow 3\theta = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \ldots$	**M1**	At least 2 of these 3 angles considered (accept degrees here) Ignore any alternatives outside $(0, \pi)$
	$\Rightarrow \theta = \frac{\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \ldots$	**A1**	
9(iii)	$2\cos 3\theta - \sqrt{3} = 0 \Rightarrow 8c^3 - 6c - \sqrt{3} = 0$	**B1**	
	Setting $x = 2\cos\theta$	**M1**	
	$\Rightarrow x = 2\cos\left(\frac{\pi}{18}\right), 2\cos\left(\frac{11\pi}{18}\right), 2\cos\left(\frac{13\pi}{18}\right)$	**A1**	Exactly these three answers (and no extras)
10(i)	$o(g_i) = 1, 2, 5$ or 10 since $o(g_i) \mid o(G)$ (by *Lagrange's Theorem*)	**B1 B1**	
10(ii)	g^0 or g^{10} = the identity (has order 1); g^5 has order 2; g^2, g^4, g^6, g^8 have order 5; g, g^3, g^7, g^9 have order 10	**B1 B1** **B1 B1**	For sets of elements with correct orders. Give B1 for all ten elements listed with no orders ✓; + B1 for $\geqslant 5$ orders ✓
10(iii)(a)	(0, 0) has order 1 (1, 0) has order 2 (0, 1), (0, 2), (0, 3), (0, 4) have order 5 (1, 1), (1, 2), (1, 3), (1, 4) have order 10	**B1**	For all ten elements (and no extras)
		M1	For at least five correct orders
		A1	All ten orders ✓
10(iii)(b)	$G_1 \cong G_2$ since … … elements can be matched by orders (valid for groups of small order) … both groups are cyclic (having an element of order 10)	**E1**	Correct answer with valid reason

Question	Answer	Marks	Guidance
11(a)(i)	$27\mathbf{A} = \begin{pmatrix} 459 & 324 \\ 324 & 270 \end{pmatrix}$ and $\mathbf{A}^2 = \begin{pmatrix} 433 & 324 \\ 324 & 244 \end{pmatrix} \Rightarrow n = 26$	**M1 A1**	For reasonable attempts at both; correct n
11(a)(ii)	$27\mathbf{A} - \mathbf{A}^2 = 26\mathbf{I}$ pre- or post-multiplied by $\mathbf{A}^{-1}$	**M1**	
	$\Rightarrow 27\mathbf{I} - \mathbf{A} = 26\mathbf{A}^{-1}$ and so $\mathbf{A}^{-1} = \dfrac{27}{26}\mathbf{I} - \dfrac{1}{26}\mathbf{A}$	**A1**	**FT** n if appropriate **SC B1** for $\mathbf{A}^{-1}$ found otherwise but still in correct form **M1 A0** for e.g. $\mathbf{A}(27 - \mathbf{A}) = \ldots$ and correct answer **M0** for e.g. dividing by a matrix
11(b)(i)	$k = \det(\mathbf{A}) = 170 - 144 = 26$	**M1 A1**	
11(b)(ii)	$\begin{pmatrix} 17 & 12 \\ 12 & 10 \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} (17+12m)x \\ (12+10m)x \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$	**B1**	
	Require $y' = mx'$ also; i.e. $12 + 10m = 17m + 12m^2$	**M1**	
	Solving a three-term quadratic: $0 = 12m^2 + 7m - 12 = (4m - 3)(3m + 4)$	**M1**	
	Since $m > 0$, $m = \dfrac{3}{4}$	**A1**	
	ALT. (i) $\begin{pmatrix} 17 & 12 \\ 12 & 10 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 17 & 29 & 12 \\ 0 & 12 & 22 & 10 \end{pmatrix}$ Transforming the unit square k = area of image //gm. = 26  	**M1 A1**	
	(ii) Then $\begin{pmatrix} 17 & 12 \\ 12 & 10 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 26\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 26 & 0 \\ 0 & 26 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ **B1** $\Rightarrow \begin{pmatrix} -9 & 12 \\ 12 & -16 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -3x + 4y = 0$ (twice); so $y = \dfrac{3}{4}x$ and $m = \dfrac{3}{4}$	**M1 M1 A1**	

Question	Answer	Marks	Guidance
12(i)	$y = \frac{1}{4}x^2 - \frac{1}{2}\ln x \Rightarrow \frac{dy}{dx} = \frac{x}{2} - \frac{1}{2x}$	**B1**	
	$1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{x}{2} - \frac{1}{2x}\right)^2$	**M1**	Attempted
		A1	Must be in the form of a perfect square; here or later
	$L = \int_2^8 \left(\frac{1}{2}x + \frac{1}{2}x^{-1}\right) dx = \left[\frac{1}{4}x^2 + \frac{1}{2}\ln x\right]_2^8$	**M1**	Use of arc-length formula and attempt to integrate
	$= 15 + \ln 2$	**A1**	Or exact equivalent

Question	Answer	Marks	Guidance
12(ii)	$S = 2\pi\int_2^8 \left(\frac{1}{4}x^2 - \frac{1}{2}\ln x\right)\left(\frac{1}{2}x + \frac{1}{2}x^{-1}\right)\mathrm{d}x$	**M1**	Attempted
	$= \frac{1}{4}\pi\int_2^8 \left(x^2 - 2\ln x\right)\left(x + \frac{1}{x}\right)\mathrm{d}x$	**A1**	All correct, unsimplified (ignore limits here)
	$= \frac{1}{4}\pi\int_2^8 \left(x^3 + x - 2x\ln x - 2\frac{1}{x}\ln x\right)\mathrm{d}x$	**A1**	In a form ready to integrate, term-by-term (ignore incorrect overall multiples)
	$\int(\ln x).x\,\mathrm{d}x = \frac{1}{2}x^2\ln x - \int\frac{1}{2}x^2.\frac{1}{x}\,\mathrm{d}x$	**M1**	Integration by parts (parts in correct order)
	$= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2$	**A1 A1**	✓ intermediate; ✓ final
	$\int(\ln x).\frac{1}{x}\,\mathrm{d}x = \frac{1}{2}(\ln x)^2$	**M1 A1**	Integration by parts (looped) or "recognition" or by substitution (e.g. $u = \ln x$)
	$S = \frac{1}{4}\pi\left[\frac{1}{4}x^4 + x^2 - x^2\ln x - (\ln x)^2\right]_2^8$	**M1**	Altogether, with limits (2, 8) substituted
	$= \frac{1}{4}\pi\left[1024 + 64 - 64\ln 8 - (\ln 8)^2 - 4 - 4 + 4\ln 2 + (\ln 2)^2\right]$ $= \pi\left(270 - 47\ln 2 - 2(\ln 2)^2\right)$	**A1**	**Given Answer** legitimately shown from use of $\ln 8 = 3\ln 2$.

Question	Answer	Marks	Guidance
13(i)	Π_1 has $d = \begin{pmatrix} 0 \\ -9 \\ 13 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = 44$ and Π_2 has $d = \begin{pmatrix} 8 \\ 7 \\ -3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = 28$	**M1 A1** **A1**	i.e. Π_1 is $\mathbf{r} \bullet \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = -44$ and Π_2 is $\mathbf{r} \bullet \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = -28$ Okay if d's are the negatives of these since on LHS of eqn.
	$\overrightarrow{AB} = \begin{pmatrix} 8 \\ 16 \\ -16 \end{pmatrix} = 8\mathbf{n}$ (and hence AB is // to $\mathbf{n}$)	**B1**	
13(ii)	Distance betn. planes is $\frac{1}{\lvert \mathbf{n} \rvert}(28 - -44) = 24$ since $\lvert \mathbf{n} \rvert = 3$	**M1 A1**	**Or** $DBP = \lvert \overrightarrow{AB} \rvert$ **=** $\sqrt{8^2 + 16^2 + 16^2} = 24$
13(iii)	Any two vectors perpr. to $\mathbf{n}$ e.g. $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, etc.	**B1 B1**	No 2nd B1 if one vector is a multiple of the other
	Plane Π_3 is $\mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ 5 \end{pmatrix} + \lambda\mathbf{v} + \mu\mathbf{w}$	**M1 A1**	Preferably involving U = midpoint AB but check for other possible points in Π_3; but **v**, **w** must be their chosen perpr. vectors to **n** Give **A0** if no **r** = …
13(iv)(a)	Locus is a circle	**M1**	
	in the plane Π_3	**A1**	
	centre **u**	**A1**	**FT** their **u** (can be described)
	radius 12	**A1**	

Question	Answer	Marks	Guidance
13(b)	For Π_3 of the form $\mathbf{r} = \begin{pmatrix}4\\-1\\5\end{pmatrix} + \lambda\begin{pmatrix}0\\1\\1\end{pmatrix} + \mu\begin{pmatrix}2\\0\\1\end{pmatrix}$ (e.g.) $\overrightarrow{UP} = \begin{pmatrix}2\mu\\\lambda\\\lambda+\mu\end{pmatrix}$	B1	
	Require $4\mu^2 + \lambda^2 + \lambda^2 + 2\lambda\mu + \mu^2 = 144$ i.e. $5\mu^2 + 2\lambda^2 + 2\lambda\mu = 144$	M1	
	Setting $\lambda = \mu \Rightarrow 9\mu^2 = 144$ and $\lambda = \mu = \pm 4$, giving	M1	
	$P = (12, 3, 13)$ from $\lambda = \mu = -4$ or $P = (-4, -5, -3)$ from $\lambda = \mu = 4$	A1	
	ALT. I Expressing $5\mu^2 + 2\lambda^2 + 2\lambda\mu = 144$ as a quadratic (in λ, say) gives $2\lambda^2 + 2\mu\lambda + 5\mu^2 - 144 = 0$ **B1** For *rational* solutions, its discriminant $\Delta = 4\mu^2 - 8(5\mu^2 - 144) = 36(32 - \mu^2)$ must be a perfect square **M1** This only happens for integer μ when $\mu = \pm 4$; each value of μ gives two values of λ : **M1** $\mu = 4 \Rightarrow \lambda = 4$ or -8 giving (12, 3, 13) or (12, –9, 1) or $\mu = -4 \Rightarrow \lambda = 8$ or -4 giving (–4, 7, 9) or (–4, –5, –3) **A1** (any one)		
	ALT. II For Π_3 of the form $\mathbf{r} = \begin{pmatrix}4\\-1\\5\end{pmatrix} + \lambda\mathbf{v} + \mu\begin{pmatrix}2\\1\\2\end{pmatrix}$ we already know that $\begin{pmatrix}2\\1\\2\end{pmatrix}$ has magnitude 3 **B1** So take $\lambda = 0$ and $\mu = 4$ or -4 to get $\mathbf{p} = \begin{pmatrix}4\\-1\\5\end{pmatrix} + 4\begin{pmatrix}2\\1\\2\end{pmatrix} = \begin{pmatrix}12\\3\\13\end{pmatrix}$ or $\mathbf{p} = \begin{pmatrix}4\\-1\\5\end{pmatrix} + 4\begin{pmatrix}2\\1\\2\end{pmatrix} = \begin{pmatrix}-4\\-5\\-3\end{pmatrix}$ **M1 M1 A1**		
	ALT. III For Π_3 of the form $\mathbf{r} = \begin{pmatrix}4\\-1\\5\end{pmatrix} + \lambda\begin{pmatrix}0\\1\\1\end{pmatrix} + \mu\begin{pmatrix}2\\0\\1\end{pmatrix}$ (e.g.) $\mathbf{p} = \begin{pmatrix}4+2\mu\\-1+\lambda\\5+\lambda+\mu\end{pmatrix}$ **B1** Either AP or $BP = 12\sqrt{2}$ (and squaring) $\Rightarrow 5\mu^2 + 2\lambda^2 + 2\lambda\mu = 144$ (again) **M1** For integer solutions, μ must be even: $\mu = 0 \Rightarrow \lambda^2 = 72$; $\mu = \pm 2 \Rightarrow (\lambda \pm 1)^2 = 63$; $\mu = \pm 4 \Rightarrow (\lambda \pm 2)^2 = 36$, which gives viable solutions: $\lambda \pm 2 = 6$ or -6 **M1** i.e. $\mu = 4$, $\lambda = 4$ or -8 giving (12, 3, 13) or (12, –9, 1) or $\mu = -4$, $\lambda = 8$ or -4 giving (–4, 7, 9) or (–4, –5, –3) **A1**		

Question	Answer	Marks	Guidance
13(b)	**ALT. IV** For Π_3 in the form $\mathbf{r} \bullet \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = -8$, we could (e.g.) write P with position vector $p = \begin{pmatrix} 2a-2b \\ b-4 \\ a \end{pmatrix}$ so that $\overrightarrow{AP} = \begin{pmatrix} 2a-2b \\ b+5 \\ a-13 \end{pmatrix}$ Then $AP^2 = 288 \Rightarrow (2a-2b)^2 + (b+5)^2 + (a-13)^2 = 288 \Leftrightarrow 5(a-5)^2 + 5(b-3)^2 - 8(a-5)(b-3) = 144$ c.f. $5(x^2+y^2) = 144 + 8xy$. Noting that LHS $\geqslant 0$, and a multiple of 5, and RHS a multiple of 8, we have		

$144 + 8xy =$	40	8	120	160	200	240	280	320	360	400	440	480
when $xy =$	–13	–8	–3	2	7	12	17	22	27	32	37	42
and $x^2 + y^2 =$	8	16	24	32	40	48	56	64	72	80	88	96
$(x, y) =$	×	×	×	×	×	×	×	×	×	$(\pm4, \pm8)$ or v.v.	×	×

For the one viable case, x and y have the same sign; so there are 4 solutions … giving (12, 3, 13), (12, –9, 1), (–4, 7, 9) and (–4, –5, –3)