Differentiation

Question Paper

Level	Pre U
Subject	Maths
Exam Board	Cambridge International Examinations
Topic	Differentiation
Booklet	Question Paper

Time Allowed: **175 minutes**

Score: **/146**

Percentage: **/100**

Grade Boundaries:

1 The equation of a curve is $y = x^3 - 2x^2 - 4x + 3$.

(i) Find $\frac{dy}{dx}$. [2]

(ii) Hence find the coordinates of the stationary points on the curve. [4]

2 The parametric equations of a curve are

$$x = e^{2t} - 5t, \quad y = e^{2t} - 3t.$$

(i) Find $\frac{dy}{dx}$ in terms of t. [3

(ii) Find the equation of the tangent to the curve at the point when $t = 0$, giving your answer in the form $ay + bx + c = 0$ where a, b and c are integers. [5]

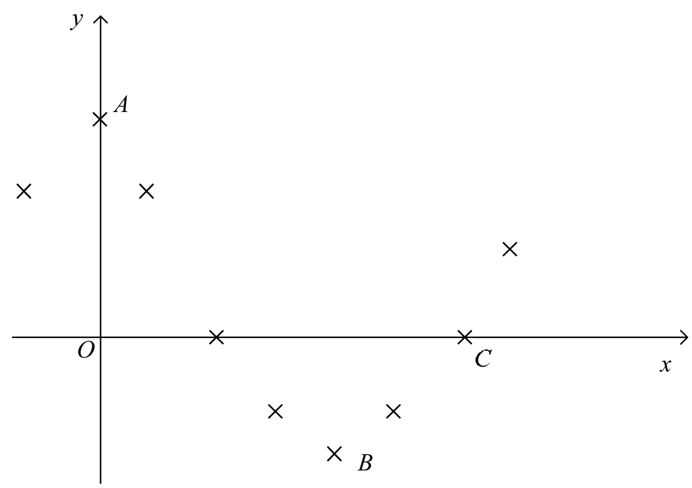
3 A curve has equation

$$y = e^{ax} \cos bx$$

where a and b are constants.

(i) Show that, at any stationary points on the curve, $\tan bx = \frac{a}{b}$. [4]

(ii)



Values of related quantities x and y were measured in an experiment and plotted on a graph of y against x, as shown in the diagram. Two of the points, labelled A and B, have coordinates (0, 1) and (0.2, –0.8) respectively. A third point labelled C has coordinates (0.3, 0.04). Attempts were then made to find the equation of a curve which fitted closely to these three points, and two models were proposed.

In the first model the equation is $y = e^{-x} \cos 15x$.

In the second model the equation is $y = f \cos(\lambda x) + g$, where the constants f, λ, and g are chosen to give a maximum precisely at the point $A(0, 1)$ and a minimum precisely at the point $B(0.2, -0.8)$.

By calculating suitable values evaluate the suitability of the two models. [12]

4 The parametric equations of a curve are given by

$$x = e^t - 2t, \quad y = e^t - 5t.$$

(i) Find $\frac{dy}{dx}$ in terms of t. [2]

(ii) Show that $t = -\ln 2$ at the point on the curve where the gradient is 3. [4]

5 Given that $f(x) = x^3$, use differentiation from f rst principles to prove that $f'(x) = 3x^2$. [4]

6 Show that the graph of $y = x^2 - \ln x$ has only one stationary point and give the coordinates of that point in exact form. [6]

7 The cubic equation $x^3 - 2x^2 + 4x - 7 = 0$ has a single root α, close to 1.9, which can be found using an iteration of the form $x_{n+1} = \mathrm{F}(x_n)$. Three possible functions that can be used for such an iteration are

$$\mathrm{F}_1(x) = \tfrac{7}{4} + \tfrac{1}{2}x^2 - \tfrac{1}{4}x^3, \qquad \mathrm{F}_2(x) = \sqrt[3]{2x^2 - 4x + 7}, \qquad \mathrm{F}_3(x) = \frac{7 - 4x}{x^2 - 2x}.$$

(i) Differentiate each of these functions with respect to x. [5]

(ii) Without performing any iterations, and using $x = 1.9$, show that an iterative process based on only two of the given functions will converge.

Determine which one will do so more rapidly. [4]

The sequence of errors, e_n, is such that $e_{n+1} \approx \mathrm{F}'(\alpha)e_n$.

(iii) Using the iteration from part **(ii)** with the most rapid convergence, estimate the number of iterations required to reduce the magnitude of the error from $|e_1|$ in the f rst term to less than $10^{-10}|e_1|$. [3]

8 A curve C is define parametrically by

$$x = \cos t(1 - 2\sin t), \quad y = \sin t(1 - 3\sin t), \quad 0 \leqslant t < 2\pi.$$

(i) Show that C intersects the y-axis at exactly three points, and state the values of t and y at these points. [5]

(ii) Find the range of values of t for which C lies above the x-axis. [4]

9 A curve has parametric equations given by

$$x = 2\sin\theta, \quad y = \cos 2\theta.$$

(i) Show that $\dfrac{dy}{dx} = -2\sin\theta$. [4]

(ii) Hence find the equation of the tangent to the curve at $\theta = \frac{1}{2}\pi$. [3]

(iii) Find the cartesian equation of the curve. [3]

10 The curve C has equation $x^2 + xy + y^2 = 19$.

(i) Show that $\dfrac{dy}{dx} = \dfrac{-2x - y}{x + 2y}$. [4]

(ii) Hence find the equation of the normal to C at the point $(2, 3)$ in the form $ax + by + c = 0$, where a, b and c are integers. [4]

11 It is given that $y = x^2 e^{-x}$.

(i) Show that $\dfrac{dy}{dx} = xe^{-x}(2 - x)$. [4

(ii) Hence find the exact coordinates of the stationary points on the curve $y = x^2 e^{-x}$. [3]

12 The equation of a curve is $y = x^3 + x^2 - x + 3$.

(i) Find $\dfrac{dy}{dx}$. [2]

(ii) Hence find the coordinates of the stationary points on the curve. [4]

13 Let $y = (2x - 3)e^{-2x}$.

(i) Find $\frac{dy}{dx}$, giving your answer in the form $e^{-2x}(ax + b)$, where a and b are integers. [3]

(ii) Determine the set of values of x for which y is increasing. [2]

14 **(i)** A curve C_1 is defined by the parametric equations

$$x = \theta - \sin\theta, \quad y = 1 - \cos\theta,$$

where the parameter θ is measured in radians.

(a) Show that $\frac{dy}{dx} = \cot\frac{1}{2}\theta$, except for certain values of θ, which should be identified. [5]

(b) Show that the points of intersection of the curve C_1 and the line $y = x$ are determined by an equation of the form $\theta = 1 + A\sin(\theta - \alpha)$, where A and α are constants to be found, such that $A > 0$ and $0 < \alpha < \frac{1}{2}\pi$. [4]

(c) Show that the equation found in part **(b)** has a root between $\frac{1}{2}\pi$ and π. [2]

(ii) A curve C_2 is defined by the parametric equations

$$x = \theta - \tfrac{1}{2}\sin\theta, \quad y = 1 - \tfrac{1}{2}\cos\theta,$$

where the parameter θ is measured in radians. Find the y-coordinates of all points on C_2 for which $\frac{d^2y}{dx^2} = 0$. [4]

15 The parametric equations of a curve are $x = \frac{1}{1 + t^2}$ and $y = \frac{t}{1 + t^2}$, $t \in \mathbb{R}$.

(i) Find $\frac{dy}{dx}$ in terms of t. [5]

(ii) Hence find the coordinates of the stationary points of the curve. [2]

16 A curve has equation $x^2 - xy + y^2 = 1$.

(i) Find $\frac{dy}{dx}$ in terms of x and y. [4]

(ii) Find the coordinates of the points on the curve in the second and fourth quadrants where the tangent is parallel to $y = x$. [5]

17 Let $y = (x-1)\left(\frac{2}{x^2} + t\right)$ define y as a function of x ($x > 0$), for each value of the real parameter t.

(i) When $t = 0$,

(a) determine the set of values of x for which y is positive and an increasing function, [3]

(b) locate the stationary point of y, and determine its nature. [2]

(ii) It is given that $t = 2$ and $y = -2$.

(a) Show that x satisfies $f(x) = 0$, where $f(x) = x^3 + x - 1$. [1]

(b) Prove that f has no stationary points. [2]

(c) Use the Newton-Raphson method, with $x_0 = 1$, to find x correct to 4 significant figures. [4]